Midterm 2

Name:

This test has **three** questions and four numbered pages. Write your name at the top of each page. Answer on the question paper; you may write on both sides.

You may use any result that was proved or stated in lectures without giving the proof, as long as you state the result clearly. (It is not necessary to state the theorem number.) For full marks, you must justify your answers.

You may assume without proof that the following languages are **NP**-complete:

- SAT, CNF-SAT, 3-SAT and 3-COL.
- INDSET: graph G, integer k; does G contain a set of k vertices with no edges between them?
- CLIQUE: graph G, integer k; does G contain a set of k vertices with all possible edges between them?
- HAMCYCLE: does a graph G contain a cycle that visits every vertex exactly once?
- TSP (traveling salesman): given n cities, distances between all pairs and target distance d, is it possible to visit every city exactly once and return to the start point while traveling distance at most d?

Name:

Question 1 (30 marks)

Say that a Turing machine M *approximates the halting problem* if for every string $\langle M' \rangle w$ that M accepts, M' halts on input w and, for every string $\langle M' \rangle w$ that M rejects, M' does not halt on input w. If M does not halt on input $\langle M' \rangle w$, this tells us nothing about whether $M'(w)$ halts. Let $L = \{\langle M \rangle \mid M \text{ approximates the halting problem}\}$.

a) Show that L is undecidable. [15 marks]

b) Is L semi-decidable? [15 marks]

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Question 2 (35 marks)

Let $K(x)$ be the Kolmogorov complexity of a string x. Recall that a string is *incompressible* if $K(x) \geq |x|$.

a) Show that there is a constant $c \geq 0$ such that, for all strings x and y, $K(xy) \leq K(x) + K(y) + 2\log|xy| + c$. [15 marks]

b) Show that there is a constant $c \geq 0$ such that, for every incompressible string x, every substring v of x has $K(v) \geq |v| - 4\log n - c$. [20 marks]

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Question 3 (35 marks)

a) If $\mathbf{P} = \mathbf{NP}$, what languages are $\mathbf{NP}$-complete? [5 marks]

b) A *vertex cover* in an undirected graph $G = (V, E)$ is a set $S \subseteq V$ such that every edge has at least one endpoint in S. Let VertexCover be the set of strings G, k such that G encodes a graph that has a vertex cover of size at most k. Prove that VertexCover is $\mathbf{NP}$-complete. [15 marks]

c) Show that the following problem is $\mathbf{NP}$-complete. You are given a directed graph G and an integer k; is it possible to produce a graph with no directed cycles by deleting at most k vertices from G? (A directed cycle is a sequence $v_1, \ldots, v_\ell$ of distinct vertices such that $\ell \geq 2$ and there are directed edges $v_1 \to v_2, \ldots, v_{\ell-1} \to v_\ell$ and $v_\ell \to v_1$.) [15 marks]