CS174 Final Exam Dec. 17, 2002

The number of points for each question or part of each question is given in parentheses. The total number of points is 100. Problems 5, 6b, 7b, 8 and 10b ask for a proof, explanation or description. These can be brief. In the other problems you only need to write down the answer, which should be clearly marked by enclosing it in a box. If you show your work in the space provided you may receive partial credit even if your answer is incorrect. If part of your solution is on a back page indicate so with an arrow or the word 'over.'

Your name:

1. (5) Let $X_1, \cdots, X_n$ be a sequence of independent binary random variables, with each X_i being equal to 1 with probability p. A maximal consecutive sequence of 1's is called a *run*. For instance, the sequence $1, 0, 1, 1, 1, 0, 0, 1, 1, 1, 0$ contains 3 runs. What is the expected number of runs in the sequence $X_1, \cdots, X_n$?

   $p + (n-1)p(1-p)$

2. (8) Let random variable X be the number of tosses of a coin with probability of heads p, up to and including the first head.

   (a) (2)What is the expectation of X? $\frac{1}{p}$

   (b) (3) Let Y and Z be independent random variables, each with the same distribution as X. What is the expectation of $\min(X, Y)$? Hint: consider an experiment in which, at each step, two independent coins with probability of heads p are tossed. $\frac{1}{2p-p^2}$

   (c) (3) What is the expectation of $\max(X, Y)$?

   $\frac{1}{2p-p^2} + \frac{1}{p}$

3. (7) In a random permutation of n elements, what is the probability that elements 1 and 2 are in the same cycle? $1/2$

$A_k \stackrel{\text{def}}{=}$ event that 1 is in a cycle of length k

$B \stackrel{\text{def}}{=}$ event that 1 and 2 are in same cycle

$$P(A_k) = \frac{1}{n} \qquad P(B \mid A_k) = \frac{k-1}{n-1}$$

$$P(B) = \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{k-1}{n-1} = \frac{1}{2}$$

4. (5) Let X have the Poisson distribution with expectation a; this means that, for every nonnegative integer k, $P(X = k) = e^{-a}\frac{a^k}{k!}$. What is the variance of X? Hint: start by computing $E[X(X-1)]$.

a

$$E[X(X-1)] = \sum_{k=0}^{\infty} k(k-1)\, e^{-a} \frac{a^k}{k!}$$

$$= \sum_{k=2}^{\infty} e^{-a} a^k/(k-2)!$$

$$= a^2 \sum_{j=0}^{\infty} e^{-a} a^j/j! = a^2$$

$$\begin{aligned} \operatorname{Var}[X] &= E[X^2] - E[X]^2 \\ &= E[X(X-1)] + E[X] - E[X]^2 \\ &= a^2 + a - a^2 \\ &= a \end{aligned}$$

5. (5) Let X be a random variable with expectation E[X]> 0 and variance Var[X]. Prove: $P(X \leq 0) \leq \frac{Var[X]}{E[X]^2}$.

$$P(X \leq 0) \leq P(|X - E[X]| \geq E[X])$$
$$\leq \frac{Var[X]}{E[X^2]}$$

where the last inequality is an instance of Chebyshev's Inequality

6. (10) A group of $2n$ individuals, consisting of n married couples, are randomly arranged at a round table.

   (a) (3) What is the probability that a given husband and wife are seated next to each other? $\frac{2}{2n-1}$

   (b) (7) Using the inequality $P(X \geq 1) \geq \sum_i \frac{p_i}{E[X|X_i=1]}$, give a lower bound on the probability that some husband and wife are seated next to each other. Show your calculation.

$$X_i = \begin{cases} 1 & \text{if couple } i \text{ seated together} \\ 0 & \text{o.w} \end{cases}$$

$$P_i = P(X_i) = \frac{2}{2n-1}$$

$$X = \sum_{i=1}^{n} X_i$$

$$E[X | X_i = 1] = 1 + (n-1)\frac{2n-3}{\binom{2n-2}{2}} = 2$$

$$P(X \geq 1) \geq \sum_{i=1}^{n} \frac{P_i}{E[X|X_i]} = \frac{n\, 2/2n-1}{2} =$$

$$\frac{n}{2n-1}$$

7. (10) Consider the problem of permutation routing on the 7-dimensional hypercube, where the permutation to be routed is the negation permutation; i.e., the destination of the packet starting at node x is $\bar{x}$, where $\bar{x}$ is the bitwise negation of X. For example, the destination of the packet starting at 0010110 is 1101001. Suppose we use the deterministic bit-fixing algorithm in order to route the packets, where we fix the bits from left to right.

   (a) (3) Let v be some packet that traverses the edge from 1001011 to 1001111 at some stage of the execution. What is the original location of v?. 0110011

   (b) (7) How many steps will it take to route the negation permutation on the n-dimensional hypercube using the deterministic bit-fixing algorithm, where we fix the bits from left to right? Give a brief reason for your answer.

   n

   No two packets contend for the same edge at the same time, so each packet will reach its destination in n steps.

8. (10) Suppose you have access to two subroutines. One calculates $P(x_1, x_2, x_3, x_4)$ and the other calculates $Q(x_1, x_2, x_3, x_4)$ where P and Q are polynomials of maximum degree 5 over the field of rational numbers. You can evaluate these functions with any combination of integer values, but you have no access to the code of the subroutines. Describe a *deterministic* way to determine whether the two subroutines compute identical functions, using exactly 6^4 function evaluations. Hint: combine the Schwartz-Zippel Theorem with the Probabilistic Method.

Choose a set S of size 6, say $S = \{0, 1, 2, 3, 4, 5\}$. Try all 6^4 substitutions

$x_1 = r_1, \; x_2 = r_2, \; x_3 = r_3, \; x_4 = r_4$

where each r_i comes from S.

Schwartz-Zippel shows that, for a random such substitution,

$$P\left(P(r_1, r_2, r_3, r_4) = Q(r_1, r_2, r_3, r_4)\right) \leq \frac{d}{|S|} = \frac{5}{6} < 1 \text{ if } P \not\equiv Q.$$

Therefore, since this probability is less than 1, some such substitution must give

$P(r_1, r_2, r_3, r_4) \neq Q(r_1, r_2, r_3, r_4)$

if $P \not\equiv Q$.

9. (10) Consider the top-in-at-random card shuffle of a deck with n cards (in each step the top card is inserted at a random position in the deck). Until the kth card from the bottom reaches the top there can be no change in the relative order in the deck of the original bottom k cards, so we cannot consider the deck well shuffled. What is the expected number of steps until the kth card from the bottom reaches the top?

Let c be the k^{th} card from the bottom. When c has risen to be the t^{th} card from the bottom, the chance that it will rise at the next step is $\frac{t}{n}$. The expected number of trials until c rises from this position is n/t. The total expected number of trials is

$$\sum_{t=k}^{n-1} \frac{n}{t} = n\left(H_{n-1} - H_{k-1}\right)$$

10. (10) Let K_n be the complete graph on vertex set $\{1, 2, \cdots, n\}$, in which all pairs of vertices are adjacent. Consider a random walk on K_n.

(a) (4) Give the value of H_{12} the expected time to reach vertex 2 starting at vertex 1.

(b) (6) A unit resistor is placed in each edge of K_n. What is the effective resistance between vertices 1 and 2?

a) $n-1$

b) $2m\, R_{12} = H_{12} + H_{21}$

$$2\, \frac{n(n-1)}{2} R_{12} = 2(n-1)$$

$$R_{12} = \frac{2}{n}$$

11. (10) The *line graph* L_n has vertices $0, 1, \cdots, n$ and an edge between vertex i and vertex $i+1$, for $i = 0, 1, \cdots, n-1$. Consider a random walk on L_n. We showed that $H_{0n} = n^2$.

(a) (2) Give the value of H_{0i} as a function of i;

(b) (3) Prove: if $i < j$ then $H_{0j} = H_{0i} + H_{ij}$.

(c) (3)Give a formula for H_{ij} as a function of i and j;

(d) (2) Is it true in general that $H_{ij} = H_{ji}$?

a) i^2

b) The walk from i to j must go from 0 to i, then from i to j

c) $H_{ij} = j^2 - i^2$

d) No. For example $H_{n-1,n} = n^2 - (n-1)^2$ but $H_{n,n-1} = 1$

12. (10) Consider a branching process in which the extinction probability is s and, for each k, p_k is the probability that an individual has exactly k offspring. Let X_1 denote the number of individuals in the kth generation, and let A be the event that extinction occurs. Give formulas in terms of s and p_k for each of the following:

(a) (3) $P(A|X_1 = k)$; s^k

(b) (3) $P((X_1 = k) \wedge A)$; $p_k s^k$

(c) (4) $P(X_1 = k|A)$.

$$P(X_1 = k | A) = \frac{P((X_1 = k) \wedge A)}{P(A)} = \frac{p_k s^k}{s}$$

$$= p_k s^{k-1}$$