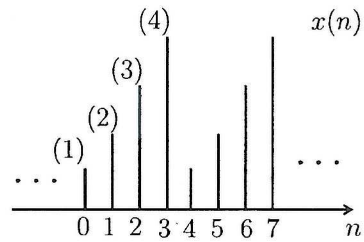
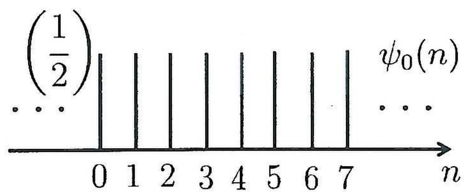
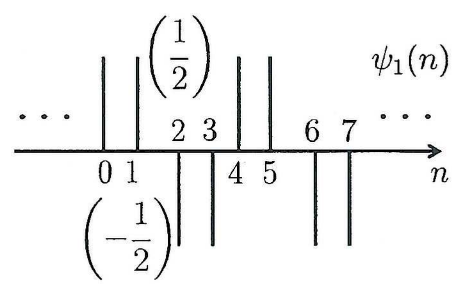
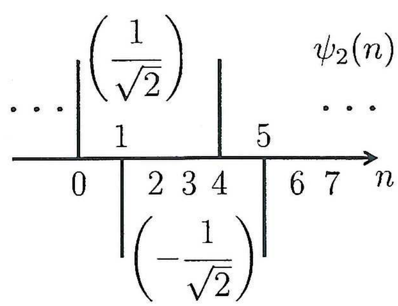
MT1.1 (40 Points) Consider a discrete-time signal x having period $p = 4$.

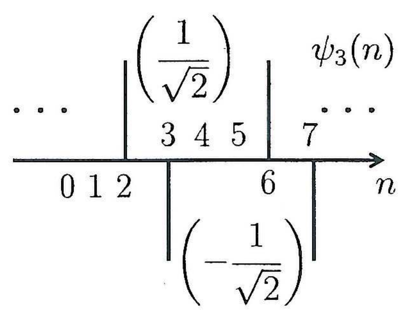


We want to express x as a linear combination of signals $\{\psi_k\}_{k=0}^{3}$, each of period 4.









+10 (a) Show that $\langle \psi_k, \psi_\ell \rangle = \delta(k - \ell)$, for all $k, \ell \in \{0, 1, 2, 3\}$.

$$\langle \psi_k, \psi_\ell \rangle = \sum_{n=0}^{3} \psi_k(n)\, \psi_\ell^*(n) = \langle \psi_\ell, \psi_k \rangle$$

since ψ_k's are real-valued

$\langle \psi_0, \psi_0 \rangle = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = 1$

$\langle \psi_1, \psi_1 \rangle = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} + \left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right) + \left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right) = 1$

$\langle \psi_2, \psi_2 \rangle = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2 + 0^2 + 0^2 = 1$

$\langle \psi_3, \psi_3 \rangle = 0^2 + 0^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2 = 1$

$\langle \psi_1, \psi_0 \rangle = \langle \psi_0, \psi_1 \rangle = 2\cdot\frac{1}{2}\cdot\frac{1}{2} + 2\cdot\frac{1}{2}\cdot\frac{-1}{2} = 0$

$\langle \psi_2, \psi_0 \rangle = \langle \psi_0, \psi_2 \rangle = \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{1}{2}\cdot\frac{-1}{\sqrt{2}} = 0$

$\langle \psi_3, \psi_0 \rangle = \langle \psi_0, \psi_3 \rangle = \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{1}{2}\cdot\frac{-1}{\sqrt{2}} = 0$

$\langle \psi_1, \psi_2 \rangle = \langle \psi_2, \psi_1 \rangle = \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{1}{2}\cdot\frac{-1}{\sqrt{2}} = 0$

$\langle \psi_1, \psi_3 \rangle = \langle \psi_3, \psi_1 \rangle = \frac{-1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{-1}{2}\cdot\frac{-1}{\sqrt{2}} = 0$

$\langle \psi_2, \psi_3 \rangle = \langle \psi_3, \psi_2 \rangle = \frac{1}{\sqrt{2}}\cdot 0 + \frac{-1}{\sqrt{2}}\cdot 0 + 0\cdot\frac{1}{\sqrt{2}} + 0\cdot\frac{-1}{\sqrt{2}} = 0$

+10 (b) Determine all the coefficients X_k in the decomposition

$$x(n) = \sum_{k=0}^{3} X_k \, \psi_k(n), \quad \text{for all } n.$$

$$X_k = \frac{\langle x, \psi_k \rangle}{\langle \psi_k, \psi_k \rangle} = \langle x, \psi_k \rangle = \sum_{n=0}^{3} x(n)\psi_k^*(n)$$

$$X_0 = \tfrac{1}{2}\cdot 1 + \tfrac{1}{2}\cdot 2 + \tfrac{1}{2}\cdot 3 + \tfrac{1}{4}\cdot 4 = 5$$

$$X_1 = \tfrac{1}{2}(1+2) + \tfrac{-1}{2}(3+4) = -2$$

$$X_2 = \tfrac{1}{\sqrt{2}}\cdot 1 + \tfrac{-1}{\sqrt{2}}\cdot 2 = \tfrac{-1}{\sqrt{2}}$$

$$X_3 = \tfrac{1}{\sqrt{2}}\cdot 3 + \tfrac{-1}{\sqrt{2}}\cdot 4 = \tfrac{-1}{\sqrt{2}}$$

+15 (c) Show that, for our particular choice of signals $\{\psi_k\}_{k=0}^{3}$,

$$\sum_{n\ \cancel{k}=0}^{3} |x(n)|^2 = \sum_{k=0}^{3} |X_k|^2,$$

or $\sum_{k=0}^{3} |X_k|^2 = \sum_{n=0}^{3} |x(n)|^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$

+5 for

and evaluate $\sum_{k=0}^{3} |X_k|^2$. $= 5^2 + (-2)^2 + \left(\tfrac{-1}{\sqrt{2}}\right)^2 + \left(\tfrac{-1}{\sqrt{2}}\right)^2 = 30$

+10 for this proof

$$\langle x, x\rangle = \sum_{n=0}^{3} |x(n)|^2 = \sum_{n=0}^{3} x(n)x^*(n) = \sum_{n=0}^{3}\left(\sum_{k=0}^{3} X_k \psi_k(n)\right)\left(\sum_{l=0}^{3} X_l \psi_l(n)\right)^*$$

$$= \sum_{n=0}^{3}\left(\sum_{k=0}^{3}\sum_{l=0}^{3} X_k X_l^* \psi_k(n)\psi_l^*(n)\right) = \sum_{k=0}^{3}\sum_{l=0}^{3} X_k X_l^* \underbrace{\sum_{n=0}^{3} \psi_k(n)\psi_l^*(n)}_{\langle \psi_k, \psi_l \rangle}$$

$$= \sum_{k=0}^{3}\sum_{l=0}^{3} X_k X_l^* \delta(k-l) = \sum_{k=0}^{3} X_k X_k^* = \sum_{k=0}^{3} |X_k|^2$$

+5 (d) Evaluate $\sum_{k=0}^{3} X_k$.

$$\sum_{k=0}^{3} X_k = 5 + -2 + \tfrac{-1}{\sqrt{2}} + \tfrac{-1}{\sqrt{2}} = 3 - \tfrac{2}{\sqrt{2}} = \underline{3 - \sqrt{2}}$$

MT1.2 (40 Points) The frequency response of a causal discrete-time LTI filter H is

$$\forall \omega \in \mathbb{R}, \quad H(\omega) = \frac{1}{1 + \frac{1}{2} e^{-i\omega}}.$$

+5 (a) Determine the linear, constant-coefficient difference equation that characterizes the input-output behavior of the filter.

$$\frac{Y(\omega)}{X(\omega)} = H(\omega)$$

$$Y(\omega)\left[1 + \frac{1}{2} e^{-i\omega}\right] = X(\omega)$$

Taking the inverse FT.

$$y[n] + \frac{1}{2} y[n-1] = x(n)$$

+6 (b) Determine $h(n)$, the impulse response values of the filter.

Note that we are told the filter H is causal.

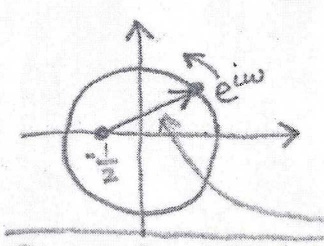
$H(\omega)$ has the standard form $\frac{1}{1 - a e^{-i\omega}}$, for which the inverse transform is $a^n u(n)$. [$u(n)$ due to causality]

$$h(n) = \left(\frac{-1}{2}\right)^n u(n)$$

+15 (c) Provide a well-labeled plot of the filter's magnitude response $|H(\omega)|$ and phase response $\angle H(\omega)$ for $|\omega| < \pi$. You must explain how you arrive at the plot.

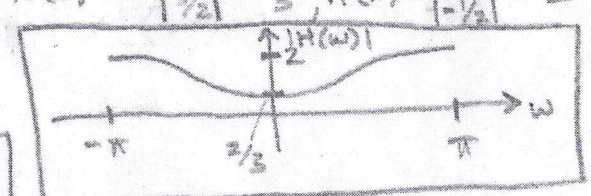
Mag. Response:

$$H(\omega) = \frac{1}{1 + \frac{1}{2} e^{-i\omega}} \cdot \frac{e^{i\omega}}{e^{i\omega}} = \frac{e^{i\omega}}{e^{i\omega} + \frac{1}{2}} \Rightarrow |H(\omega)| = \frac{|e^{i\omega}|}{|e^{i\omega} + \frac{1}{2}|} = \frac{1}{|e^{i\omega} - (-\frac{1}{2})|}$$



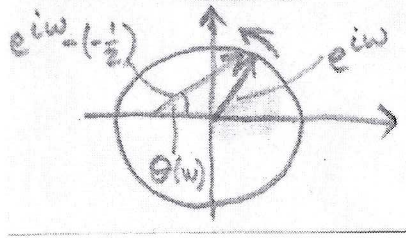
The magnitude response is 1 divided by the length of this vector.

$$\Rightarrow \quad H(0) = \frac{1}{|3/2|} = \frac{2}{3}, \quad H(\pi) = \frac{1}{|-1/2|} = 2$$

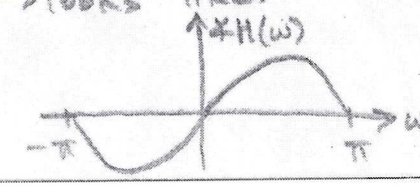


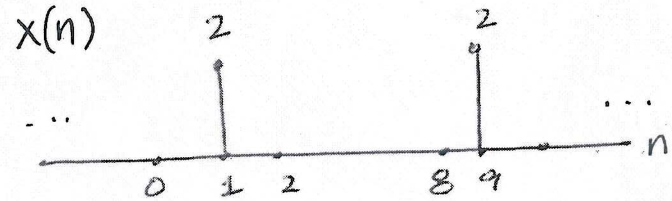
Phase Response: $\angle H(\omega) = \angle e^{i\omega} - \angle(e^{i\omega} - (-\frac{1}{2}))$

$$= \omega - \theta(\omega)$$



Qualitatively, using the above analysis, we get a phase plot that looks like:





+14 (d) We apply the following periodic discrete-time signal to the filter:

$$\forall n \in \mathbb{Z}, \quad x(n) = \sum_{\ell=-\infty}^{+\infty} 2\,\delta(n-1-8\ell).$$

+7 (i) Determine a reasonably simple expression for the discrete-time Fourier series[1] coefficients X_k of the input signal, where

$$\forall n \in \mathbb{Z}, \quad x(n) = \sum_{k=\langle 8\rangle} X_k\, e^{i2\pi k n/8}.$$

$$X_k = \frac{1}{8}\sum_{k\in\langle p\rangle} x(n)\, e^{-i\frac{2\pi k n}{8}}$$

period $p = 8$

$$= 2\cdot e^{-i\frac{2\pi k}{8}}$$

$$= \frac{1}{4}\, e^{-i\frac{\pi k}{4}}$$

+7 (ii) Express, in terms of X_k and the filter's frequency response $H(\omega)$, the discrete-time Fourier series coefficients Y_k of the corresponding output signal y. Also, find Y_4, in particular.

$$Y_k = X_k \cdot H(\omega)\Big|_{\omega=\frac{2\pi k}{p}}$$

$$= X_k\, H\!\left(\frac{2\pi k}{8}\right)$$

$$= X_k\, H\!\left(\frac{\pi k}{4}\right)$$

$$Y_4 = X_4 \cdot H(\pi)$$

$$= \left(\frac{-1}{4}\right)\cdot \frac{1}{1+\frac{1}{2}e^{-i\pi}}$$

$$= \frac{-1}{2}$$

[1] The complex exponential Fourier series synthesis and analysis equations for a periodic discrete-time signal having period p: $x(n) = \sum_{k=\langle p\rangle} X_k\, e^{ik\omega_0 n} \quad \longleftrightarrow \quad X_k = \frac{1}{p}\sum_{n=\langle p\rangle} x(n)\, e^{-ik\omega_0 n}$, where $p = \frac{2\pi}{\omega_0}$ and $\langle p\rangle$ denotes a suitable discrete interval of length p (i.e., an interval containing p contiguous integers). For example, $\sum_{k=\langle p\rangle}$ may denote $\sum_{k=0}^{p-1}$ or $\sum_{k=1}^{p}$.

Midterm 1 Problem 3 Solutions, Fall 2010

2010-09-23

The input-output behavior of a discrete-time system **F** is described below:

$$y(n) = \sum_{-\infty}^{-2n} 3x(k)$$

(a): Select the strongest correct assertion from the choices below. Explain your reasoning succinctly, but clearly and convincingly.

(i) F must be a linear system.
(ii) **F** can be a linear system.
(iii) **F** cannot be a linear system.

Proof. We will prove this directly by application of the definition of linearity.

Let $y_1(n) = \sum_{k=-\infty}^{-2n} 3x_1(k)$, $y_2(n) = \sum_{k=-\infty}^{-2n} 3x_2(k)$.

Let $\hat{x}(n) = \alpha x_1(n) + \beta x_2(n)$ for arbitrary $\alpha, \beta \in \mathbb{R}$, and $\hat{y}(n) = \sum_{k=-\infty}^{-2n} 3\hat{x}(k)$.

$$\begin{aligned}
\hat{y}(n) &= \sum_{k=-\infty}^{-2n} 3(\alpha x_1(k) + \beta x_2(k)) \\
&= \alpha \sum_{k=-\infty}^{-2n} 3x_1(k) + \beta \sum_{k=-\infty}^{-2n} 3x_2(k) \\
&= \alpha y_1(n) + \beta y_2(n)
\end{aligned}$$

Thus, the input-output function F satisfies additivity and homogeneity, therefore it is a linear function. Since we proved that the input-output equation of the system is linear, then **F** must be a linear system. $\square$

(b): Select the strongest correct assertion from the choices below. Explain your reasoning succinctly, but clearly and convincingly.

(i) **F** must be a time-invariant system.
(ii) **F** can be a time-invariant system.
(iii) F cannot be a time-invariant system.

Proof. We will prove this by contradiction.

A system is time-invariant if the time-shifted output equals the output of the system from a time-shifted input.

Assume that the system is time-invariant. Let $y(n) = \sum_{k=-\infty}^{-2n} 3x(k)$. Let $\hat{x}(n) = x(n-T)\ \forall n$, given $T \in \mathbf{Z}$. ($\hat{x}(n)$ is the input $x(n)$ delayed by time T). Let $\hat{y}(n) = \sum_{k=-\infty}^{-2n} 3\hat{x}(k)$.

$$\begin{aligned}
\hat{y}(n) &= \sum_{k=-\infty}^{-2n} 3\hat{x}(k) \\
&= \sum_{k=-\infty}^{-2n} 3x(k-T)
\end{aligned}$$

Let $u = k - T$. After the change-of-variables,

$$\hat{y}(n) = \sum_{u=-\infty}^{-2n-T} 3x(u)$$

By time-invariance,

$$\hat{y}(n) = y(n-T) = \sum_{k=-\infty}^{-2(n-T)} x(k)$$

However,

$$\sum_{u=-\infty}^{-2n-T} 3x(u) \neq \sum_{k=-\infty}^{-2(n-T)} 3x(k)$$

This contradicts our assumption that the system $\mathbf{F}$ is time-invariant, implying that the system must not be time-invariant. $\square$

(c): Select the strongest correct assertion from the choices below. Explain your reasoning succinctly, but clearly and convincingly.

(i) $\mathbf{F}$ must be a causal system.
(ii) $\mathbf{F}$ can be a causal system.
(iii) F cannot be a causal system.

Proof. We will prove this by counter-example. Suppose we are given $x_1(n) = x_2(n)\ \forall n \leq N$. The system $\mathbf{F}$ generates corresponding outputs $y_1(n)$, $y_2(n)$. Given that the system is causal, $y_1(n) = y_2(n)\ \forall n \leq N$.

Now, consider $x_1(n) = 0\ \forall n$. In part (a), we proved this system to be linear. By the ZIZO property of linearity, $y_1(n) = 0\ \forall n$. Let $x_2(n) = \delta(n)$.

Notice that

$$y_2(n) = \sum_{k=-\infty}^{-2n} \delta(k) = \begin{cases} 3 & \text{if } n \leq 0 \\ 0 & \text{else} \end{cases}$$

Notice that $x_1(n) = x_2(n)\ \forall n < 0$. Notice that $y_1(-1) = 0 \neq y_2(-1) = 3$. Therefore, the system $\mathbf{F}$ cannot be causal. $\square$

(d): Select the strongest correct assertion from the choices below. Explain your reasoning succinctly, but clearly and convincingly.

(i) $\mathbf{F}$ must be a BIBO stable system.
(ii) $\mathbf{F}$ can be a BIBO stable system.
(iii) F cannot be a BIBO stable system.

Proof. We will prove this by counter-example.

Let $x(n) = u(-n)$, the time-reversed unit step.

Observe that $|x(n)| \leq 1\ \forall n$, so $B_x = 1$ and the input signal is bounded.

$$y(0) = \sum_{k=-\infty}^{0} 3x(k) = \sum_{k=-\infty}^{0} 3$$

Note that this sum is not well-defined. There is no number $B_y \in \mathbb{R}$ such that $|y(0)| \leq B_y$. Thus, the output is unbounded.

Since we have illustrated a bounded input which yields an unbounded output from the system $\mathbf{F}$, the system cannot be BIBO stable. $\square$

Proof. Here, we provide an alternative proof, inspired by a popular response to the problem. The response attempted to use the theorem presented in class which states that an LTI system is BIBO stable if and only if $h(n)$ is absolutely summable. The direct application of this theorem is invalid, however, since in part (b), we proved that this system is not time-invariant.

Let $x(n) = \delta(n)$.

Let

$$h(n) = \sum_{k=-\infty}^{-2n} 3\delta(k) = \begin{cases} 3 & \text{if } n \leq 0 \\ 0 & \text{else} \end{cases}$$

Note that $h(n) = 3u(-n)$ is bounded ($h(n) \leq 3\ \forall n$). Now, consider a new input $\hat{x}(n) = h(n)$. The output of the system is $\hat{y}(n) = \sum_{-\infty}^{-2n} 9u(-k)$. Clearly, this is an infinite sum of constant terms (9) at every instant in time. Thus, the output of the system is not bounded. Since we have provided a bounded input that yielded an unbounded output, the system cannot be BIBO stable.

$\square$