EECS 120. Solutions to Midterm No. 1, February 17, 2000.

1. **20 points** Find the expression for the frequency response from x to y in terms of H_1, H_2, H_3 for the system depicted in:

   (a) Part (a) of Figure 1.

   Ans Fix $\omega \in Reals$ and take as input the signal $x : t \mapsto X(\omega)e^{j\omega t}$. Since H_1 is LTI, the other signals are of the form $w : t \mapsto W(\omega)e^{j\omega t}$ and $y :\mapsto Y(\omega)e^{j\omega t}$. Moreover,

   $$W(\omega) = X(\omega) + Y(\omega), \quad Y(\omega) = H_1(\omega)W(\omega),$$

   from which we obtain the desired frequency response,

   $$\boxed{H_4(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{H_1(\omega)}{1-H_1(\omega)}}$$

   (b) Part (b) of Figure 1.

   Ans Rewrite part (b) of the figure as (c), and introduce the signal names u, v. Take as input the signal $x : t \mapsto X(\omega)e^{j\omega t}$. Since H_1, H_2, H_3 are all LTI, u, v, y are all exponential,

   $$u : t \mapsto U(\omega)e^{j\omega t}, \quad v : t \mapsto V(\omega)e^{j\omega t}, \quad y : t \mapsto Y(\omega)e^{j\omega t}.$$

   From the previous part we know $V(\omega) = H_4(\omega)U(\omega)$. And then, following the same argument as in the previous part,

   $$\frac{Y(\omega)}{X(\omega)} = \frac{H_4(\omega)H_3(\omega)}{1 - H_2(\omega)H_3(\omega)H_4(\omega)}.$$

   And substituting for H_4 from the previous part,

   $$\boxed{\frac{Y(\omega)}{X(\omega)} = \frac{H_4(\omega)H_3(\omega)}{1-H_2(\omega)H_3(\omega)H_4(\omega)} = \frac{H_1(\omega)H_3(\omega)}{1-H_1(\omega)-H_1(\omega)H_2(\omega)H_3(\omega)}}$$

2. **20 points** Let f, g, x, y be as in Figure 2.

   (a) Determine $f * g$.

   Ans Since f is periodic with period 2, $f * g$ is periodic with period 2 and it can be obtained as the periodic repetition of $f_1 * g$ where $f_1(t) = f(t), 0 \le t \le 2$ and $f_1(t) = 0$, otherwise. f_1 and $f_1 * g$ are shown graphically in the figure.

   (b) Determine $x * y$.

   Ans Arguing in the same way, $x * y$ is the periodic repetition of $x_1 * y$, with period 1.5, and we can use the calculation of $f_1 * g$ as shown in the figure.

3. **20 points** Give an example of a discrete-time system H that is:

   (a) Not linear;

   Ans Take $H(x)(n) = (x(n))^2$. This is not linear since $H(2x) = 4H(x) \neq 2H(x)$.

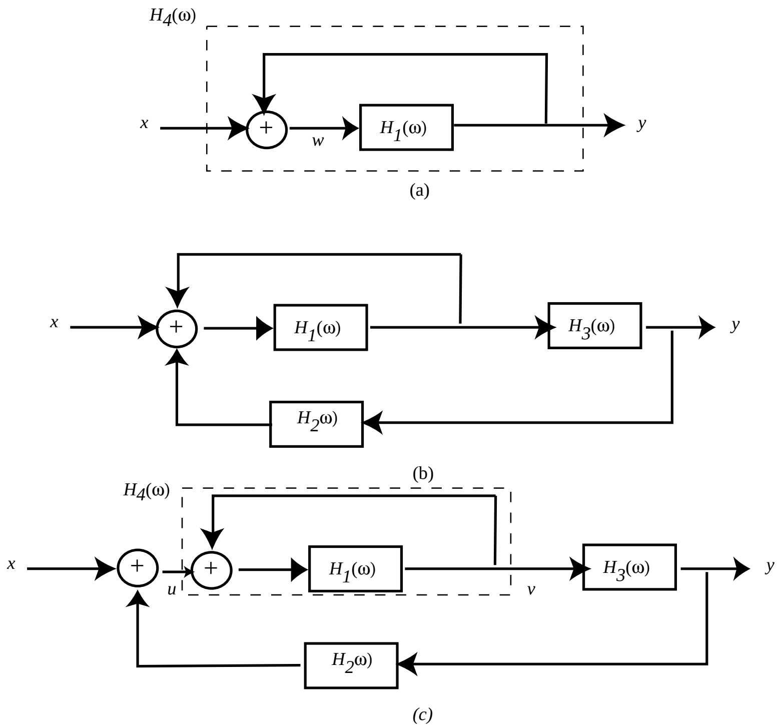

Figure 1: System for Problem 1

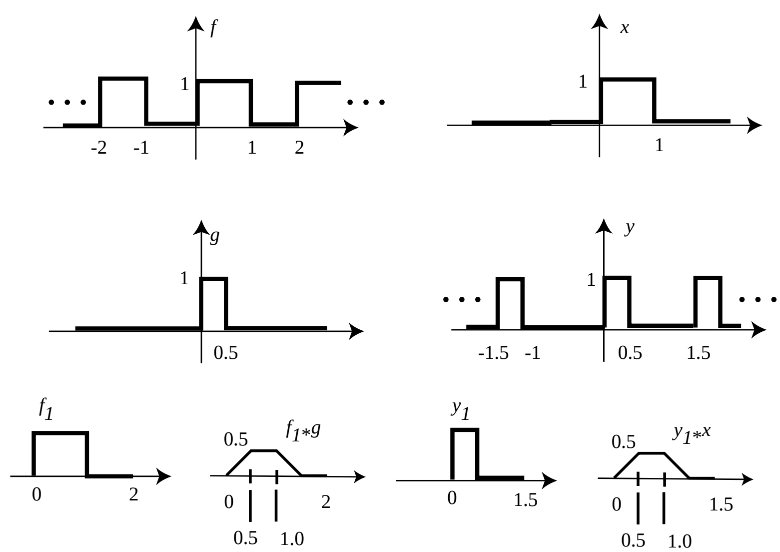

Figure 2: Signals for Problem 2

(b) Linear and time-varying;

Ans Take $H(x)(n) = x(2n)$. This is not time-invariant, since $H(D_1 x)(n) = x(2n-1) \neq D_1(Hx)(n) = H(x)(n-1) = x(2n-2)$.

(c) LTI but not causal;

Ans Take $H(x)(n) = x(n+1)$. This is not causal, because its impulse response is $n \mapsto \delta(n+1)$, so that $h(-1) = 1 \neq 0$.

(d) LTI, causal, but not memoryless.

Ans Take $H(x)(n) = x(n-1)$. This is not memoryless, because the output at n depends on the input at $n-1$.

4. **20 points** Suppose a periodic signal $x : \mathit{Reals} \to \mathit{Comps}$ with fundamental frequency ω_x has the Fourier series representation:

$$\forall t, \quad x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_x t}.$$

(a) Let y be the signal $\forall t,\ y(t) = x(t-\tau)$, where τ is a fixed number. What is the Fourier series representation of y?

Ans We have $\forall t$,

$$\begin{aligned} y(t) = x(t-\tau) &= \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_x(t-\tau)} = \sum_{k=-\infty}^{\infty} X_k e^{-j\omega_x \tau} e^{jk\omega_x t} \\ &= \sum_{k=-\infty}^{\infty} Y_k e^{jk\omega_y t}, \end{aligned}$$

where

$$\boxed{Y_k = X_k e^{-j\omega_x \tau} \text{ and } \omega_y = \omega_x.}$$

(b) Let z be the signal $\forall t,\ z(t) = x(2t)$. What is the fundamental frequency ω_z of z in terms of ω_x? What is the Fourier series representation of z?

Ans We have $\forall t$,

$$z(t) = x(2t) = \sum_{k=-\infty}^{\infty} X_k e^{jk2\omega_x t}$$

$$\sum_{k=-\infty}^{\infty} Z_k e^{jk\omega_z t}$$

where

$$\boxed{\omega_z = 2\omega_x \text{ and } Z_k = X_k.}$$

(c) Let w be the signal $\forall t,\ w(t) = z(-t)$. What is the Fourier series representation of w?

Ans We have $\forall t$,

$$\begin{aligned} w(t) = z(-t) &= \sum_{k=-\infty}^{\infty} Z_k e^{-jk\omega_z t} \\ &= \sum_{k=-\infty}^{\infty} W_k e^{jk\omega_w t} \end{aligned}$$

where

$$\boxed{\omega_w = \omega_z = 2\omega_x \text{ and } W_k = Z_{-k} = X_{-k}.}$$