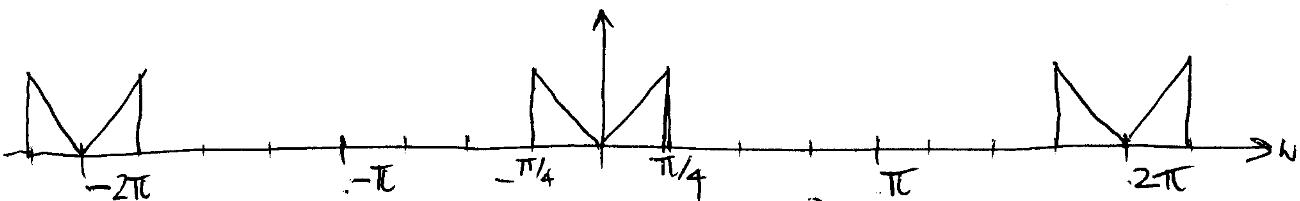
Midterm 2 Key Version A

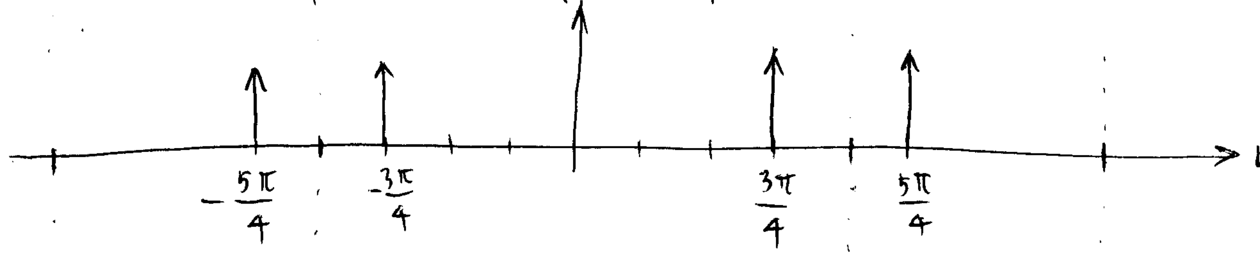
Problem 1 *(Amplitude modulation.)* 40 Points

(a) (10 Pts) For the discrete-time signal $x[n]$ it is known that $X(e^{j\omega}) = 0$, for $|\omega| > \pi/4$. Determine the range of ω for which the DTFT of $y[n] = \cos(\frac{5\pi}{4}n)x[n]$ must be zero. *Hint:* Select an example spectrum $X(e^{j\omega})$ and sketch the resulting DTFT of $y[n]$.

<u>Example spectrum:</u> $X(e^{jw})$

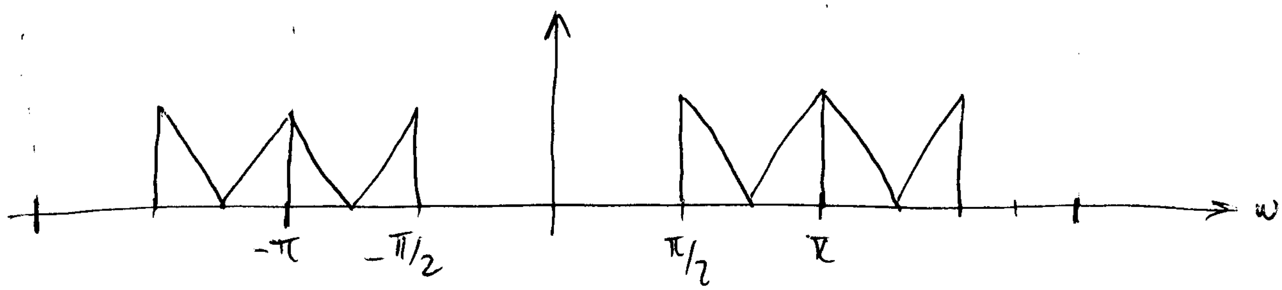


$\mathrm{DTFT}\{\cos(\frac{5\pi}{4}n)\}$



$\boxed{\text{CIRCULAR CONVOLUTION}}$ SEE O.W.N. p.389–390

$Y(e^{jw})$



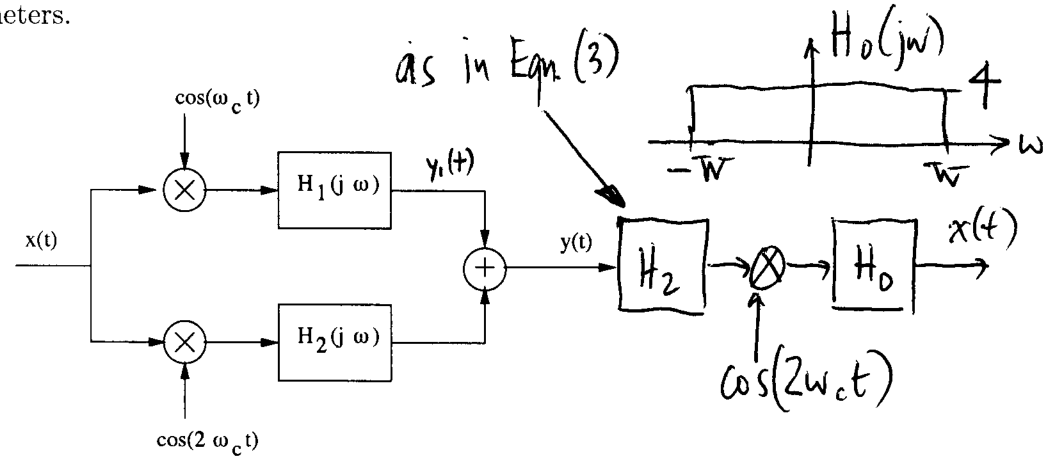
$\Rightarrow Y(e^{jw}) = 0, \quad |w| < \pi/2$

(2π - periodically repeated)

(b) (10 Pts) The real-valued data signal $x(t)$ is known to be band-limited, i.e., $X(j\omega) = 0$, for $|\omega| > W$. Consider the block diagram of Figure 1, where

$$H_1(j\omega) = \begin{cases} 1, & \text{for } |\omega| \le \omega_c \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad H_2(j\omega) = \begin{cases} 1, & \text{for } |\omega| \ge 2\omega_c \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

- Pick an arbitrary (bandlimited) example spectrum for $x(t)$, and sketch the corresponding spectrum of the signal $y(t)$.
- For what values of the parameters W and ω_c is it possible to recover $x(t)$ from $y(t)$?
- Provide the block diagram of a system that recovers $x(t)$, given $y(t)$, carefully specifying all involved parameters.



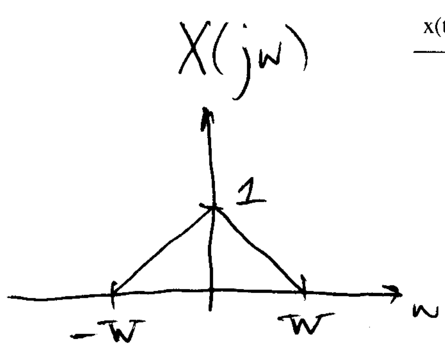
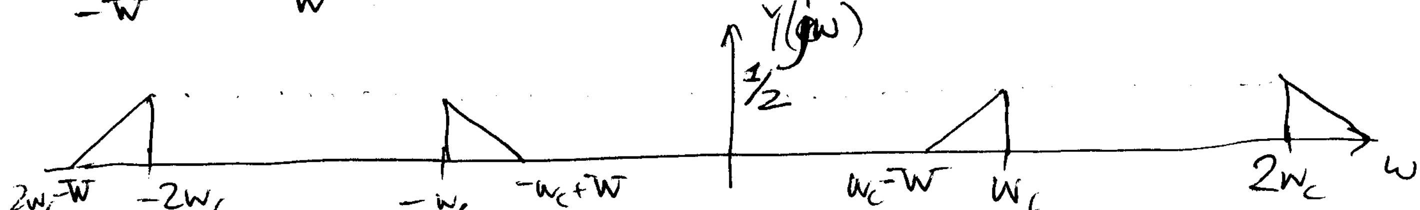
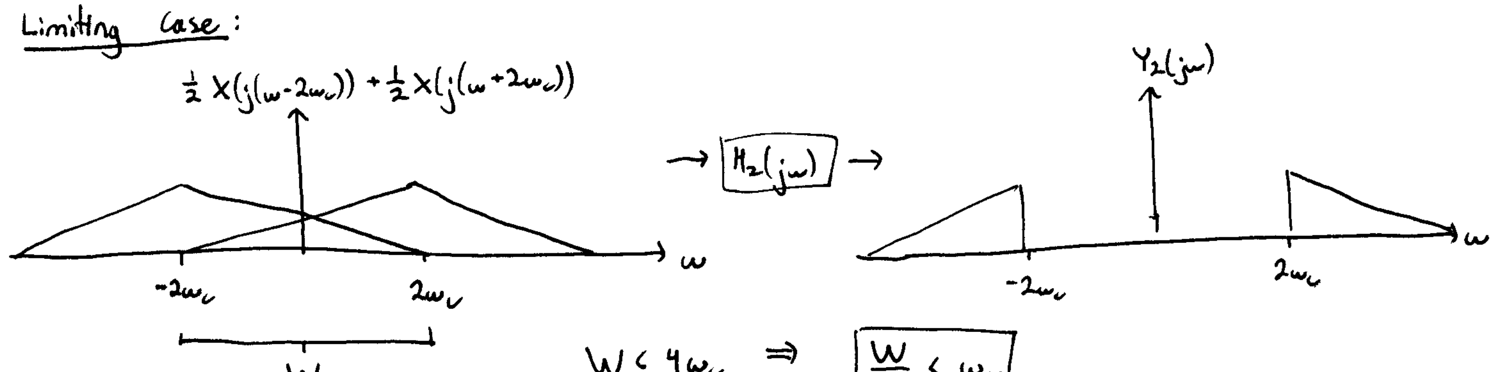


Figure 1: Block diagram for Part *(b)*.



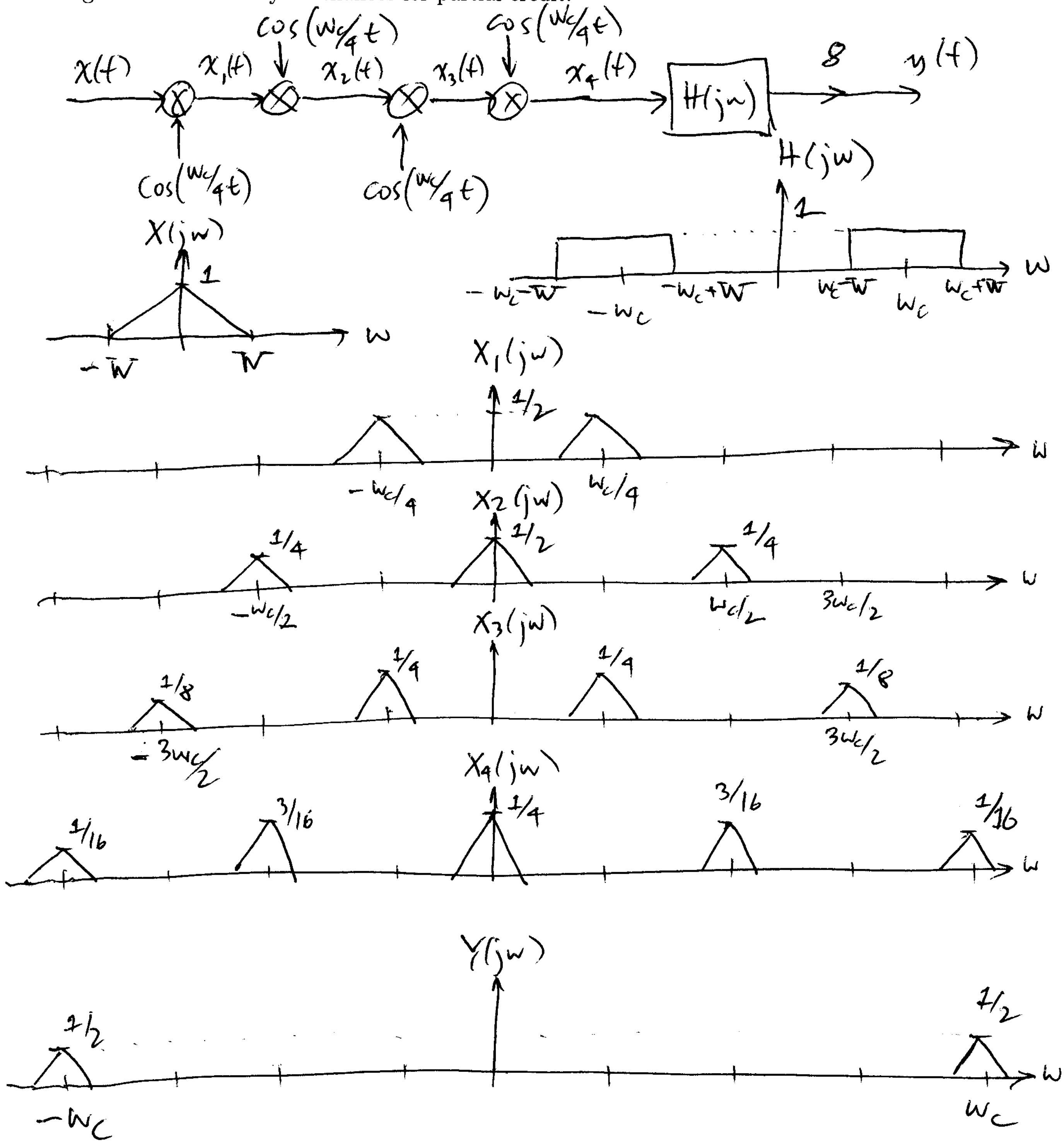
We have two copies of our signal. Therefore, even if one copy is corrupted, we can still recover our signal. Furthermore, note that the upper and lower sidebands will never interfere with one another. ($Y_1(jw) = 0 \ \forall\ |w| > w_c$ and $Y_2(jw) = 0 \ \forall\ |w| < 2w_c$)
The more generous non-aliasing condition comes from the $2w_c$ modulator.

<u>Limiting case</u>:



$W < 4w_c \Rightarrow \boxed{\frac{W}{4} < w_c}$

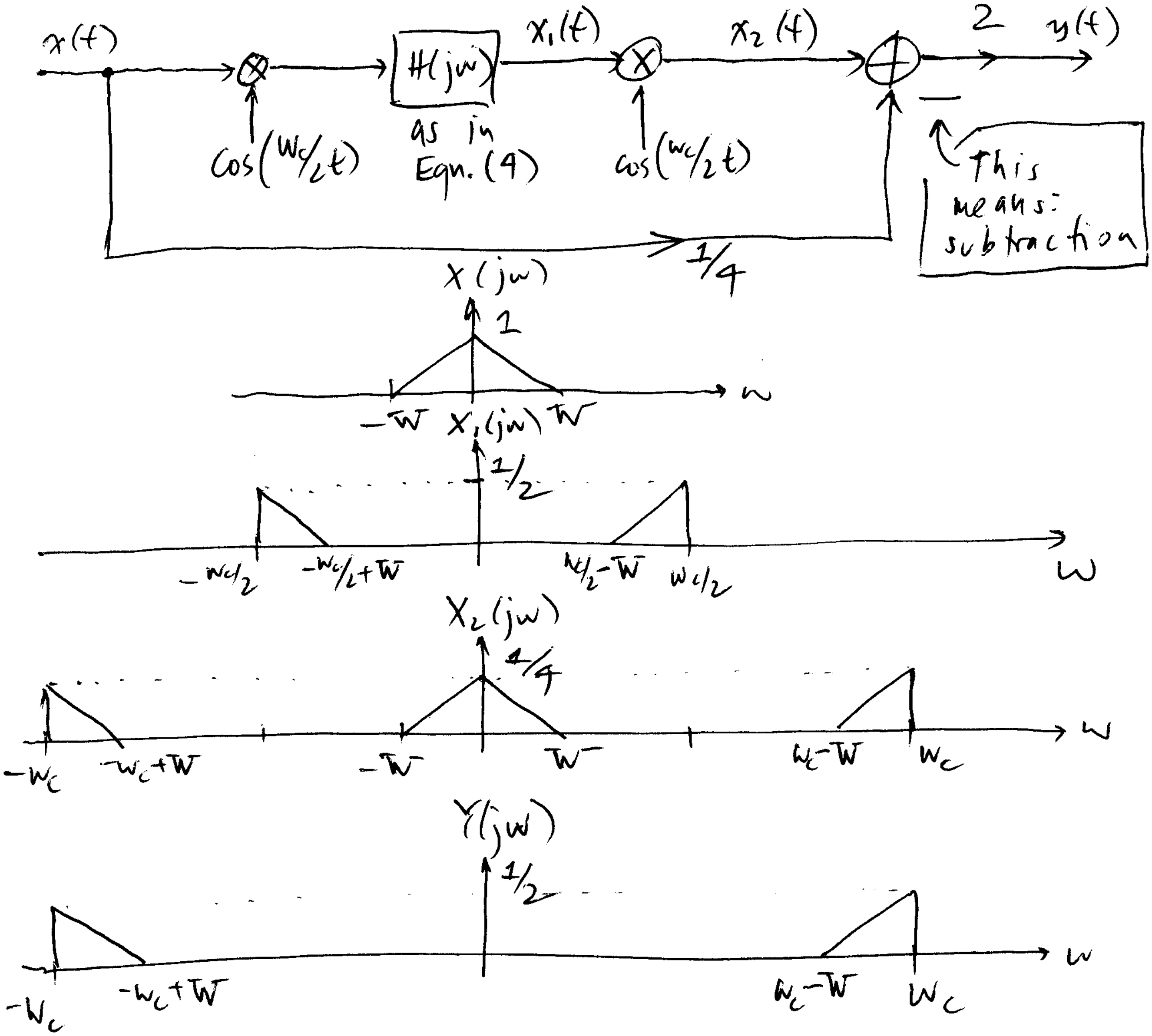
(c) (10 Pts) The real-valued data signal $x(t)$ is known to be band-limited, i.e., $X(j\omega) = 0$, for $|\omega| > W$. The goal is to perform standard (i.e., double-sideband) AM with carrier frequency $\omega_c > 5W$. Unfortunately, the only type of modulator available is multiplication by $\cos(\frac{\omega_c}{4}t)$. Otherwise, addition, scalar multiplication, and filters can be used. Draw the block diagram of the system that achieves our goal, and if your system uses a filter, specify the desired frequency response. *Hint:* Pick an example spectrum for $x(t)$ and sketch the spectra of intermediate signals to maximize your chances for partial credit.



(d) (10 Pts) The real-valued data signal $x(t)$ is known to be band-limited, i.e., $X(j\omega) = 0$, for $|\omega| > W$. The goal is to perform single-sideband AM with only the *lower* sideband, with carrier frequency $\omega_c > 5W$. Again, you can use addition, scalar multiplication, and multiplication by $\cos(\omega_m t)$, for arbitrary ω_m. However, this time, you only have *fixed* ideal low-pass filters with the following frequency response:

$$H(j\omega) = \begin{cases} 1, & \text{for } |\omega| \leq \omega_c/2 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Draw the block diagram of a system that achieves the goal, clearly specifying all involved parameters, such as the frequencies of the modulators, etc. *Hint:* Pick an example spectrum for $x(t)$ and sketch the spectra of intermediate signals to maximize your chances for partial credit.



Problem 2 *(PAM.)* 30 Points

Two pulses are suggested for a PAM system:

$$p_1(t) = ae^{-t}u(t), \quad \text{and} \quad p_2(t) = be^{-10t}u(t), \tag{5}$$

where a and b are positive real numbers that will be selected appropriately, leading to

$$y_i(t) = \sum_{k=-\infty}^{\infty} x[k]p_i(t-kT), \text{ for } i = 1, 2, \tag{6}$$

where we choose $T = 1$. We suppose that the data signal is bounded to $|x(t)| \leq 1$. In this problem, we want to compare the two pulses $p_1(t)$ and $p_2(t)$.

(a) (20 Pts) Select $a = 2$ and $b = 2\sqrt{10}$. For this choice, it can be shown that the pulse energy is the same for $p_1(t)$ and for $p_2(t)$. (You *don't* have to show this!) Now consider the transmission of $p_1(t)$ and $p_2(t)$, respectively, across a communication channel with impulse response $h(t)$ and corresponding frequency response

$$H(j\omega) = \frac{1}{6 + j\omega}. \tag{7}$$

This yields an output signal $z_i(t) = (p_i * h)(t)$, for $i = 1, 2$.

- Evaluate the energy of the received signals, $z_1(t)$ and $z_2(t)$, respectively.
- Which received signal has the larger energy?
- How is it possible that even though the two pulses have the same transmitted energy, their received energy differs?

$$E_i = \int_{-\infty}^{\infty} |z_i(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Z_i(j\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |P_i(j\omega)|^2 \, |H(j\omega)|^2 d\omega$$

$\boxed{\text{$z_1(t)$ has more of its energy concentrated in the lower frequencies. The channel is a lowpass filter so low frequencies pass through it better.}}$

$$E_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{2}{1+j\omega}\right|^2 \left|\frac{1}{6+j\omega}\right|^2 d\omega \qquad E_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{2\sqrt{10}}{10+j\omega}\right|^2 \left|\frac{1}{6+j\omega}\right|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{1^2+\omega^2}\,\frac{1}{6^2+\omega^2}\, d\omega \qquad = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{40}{10^2+\omega^2}\,\frac{1}{6^2+\omega^2}\, d\omega$$

$$= \frac{1}{2\pi}\,\frac{4\pi}{1\cdot 6(1+6)} = \boxed{\frac{1}{21}} \qquad = \frac{1}{2\pi}\cdot\frac{40\pi}{10\cdot 6(10+6)} = \boxed{\frac{1}{48}}$$

$\boxed{E_1 \text{ is larger.}}$

$$P_1(j\omega) = \frac{a}{1+j\omega} = \frac{2}{1+j\omega} \quad \text{since } a=2$$

Recall $\left|\frac{1}{c+j\omega}\right|^2 = \frac{1}{c+j\omega}\,\frac{1}{c-j\omega}$

$c \in \mathbb{R}_+$ $= \frac{1}{c^2+\omega^2}$

$$P_2(j\omega) = \frac{b}{10+j\omega} = \frac{2\sqrt{10}}{10+j\omega} \quad \text{since } b = 2\sqrt{10}$$

(b) *(10 Pts) (Hard problem)* To have a fair comparison, we have to make sure that the powers of the transmitted signals $y_1(t)$ and $y_2(t)$, respectively, are equal. To adjust the power, assume that $x[n] = 1$ for all n, i.e., for $-\infty < n < \infty$. Determine the relationship between a and b such that for this particular $x[n]$, the signals $y_1(t)$ and $y_2(t)$ have the same power. (As seen in class, this provides a worst case analysis.) *Hint:* By contrast to Part *(a)*, this question studies the *power* of the entire signal, rather than the energy of a single pulse.

$$y_1(t) = \sum_{k=-\infty}^{\infty} 1 \cdot p_1(t-kT) \underset{T=1}{=} \sum_{k=-\infty}^{\infty} p_1(t-k) = \underbrace{\sum_{k=-\infty}^{\infty} a e^{-(t-k)} u(t-k)}_{\text{Periodic with Period}=1.} = a e^{-t} \sum_{k=-\infty}^{\infty} e^{k} u(t-k)$$

Since $y_1(t)$ is periodic, $P_1 = \frac{1}{T}\int_0^T |y_1(t)|^2 dt = \int_0^1 |y_1(t)|^2 dt.$

Find a simpler form for $y_1(t)$ from $t=0$ to $t=1$. $y_1(t) \underset{0\le t\le 1}{=} a e^{-t} \sum_{k=-\infty}^{0} e^{k} = a e^{-t} \sum_{k=0}^{\infty} e^{-k}$

$y_1(t) = \frac{a e^{-t}}{1-e^{-1}}$ from $t=0$ to $t=1$.

$$P_1 = \int_0^1 \left|\frac{a e^{-t}}{1-e^{-1}}\right|^2 dt = \int_0^1 \frac{a^2 e^{-2t}}{(1-e^{-1})^2} dt = \frac{-1 \cdot a^2}{2(1-e^{-1})^2} e^{-2t}\Big|_0^1$$

$$P_1 = \frac{-a^2(e^{-2}-1)}{2(1-e^{-1})^2} = \boxed{\frac{a^2(1-e^{-2})}{2(1-e^{-1})^2}}$$

$$y_2(t) = \sum_{k=-\infty}^{\infty} 1 \cdot p_2(t-kT) \underset{T=1}{=} \sum_{k=-\infty}^{\infty} p_2(t-k) = \underbrace{\sum_{k=-\infty}^{\infty} b e^{-10(t-k)} u(t-k)}_{\text{Periodic with period 1.}} = b e^{-10t} \sum_{k=-\infty}^{\infty} e^{10k} u(t-k)$$

Since $y_2(t)$ is periodic, $P_2 = \frac{1}{T}\int_0^T |y_2(t)|^2 dt = \int_0^1 |y_2(t)|^2 dt$

Find a simpler form for $y_2(t)$ from $t=0$ to $t=1$

$y_2(t) = b e^{-10t} \sum_{k=-\infty}^{0} e^{10k} \quad 0 \le t \le 1$

$y_2(t) = b e^{-10t} \sum_{k=0}^{\infty} e^{-10k} \quad 0 \le t \le 1$

$y_2(t) = \frac{b e^{-10t}}{1-e^{-10}} \quad 0 \le t \le 1$

$$P_2 = \int_0^1 \left|\frac{b e^{-10t}}{1-e^{-10}}\right|^2 dt = \int_0^1 \frac{b^2 e^{-20t}}{(1-e^{-10})^2} dt = \frac{-1}{20} \frac{b^2 e^{-20t}}{(1-e^{-10})^2}\Big|_0^1$$

$$P_2 = \frac{-1}{20} \frac{b^2(e^{-20}-1)}{(1-e^{-10})^2} = \boxed{\frac{b^2(1-e^{-20})}{20(1-e^{-10})^2}}$$

$$P_1 = P_2 \Rightarrow \frac{a^2(1-e^{-2})}{2(1-e^{-1})^2} = \frac{b^2(1-e^{-20})}{20(1-e^{-10})^2} \Rightarrow a^2 = b^2 \frac{2}{20} \frac{(1-e^{-20})(1-e^{-1})^2}{(1-e^{-2})(1-e^{-10})^2}$$

$$\boxed{a = \frac{b}{\sqrt{10}} \sqrt{\frac{1-e^{-20}}{1-e^{-2}}} \frac{(1-e^{-1})}{(1-e^{-10})}}$$

Problem 3 *(Sampling.)* 30 Points

A real-valued data signal $x(t)$ is known to be band-limited, i.e., $X(j\omega) = 0$, for $|\omega| > W$.

(a) (12 Pts) Suppose the signal $x(t)$ is sampled non-uniformly using the impulse train $q_1(t)$ shown in Figure 2. Show that the spectrum of the sampled signal $y_1(t) = x(t)q_1(t)$ is

$$Y_1(j\omega) \quad = \quad \frac{1}{2T}\sum_{k=-\infty}^{\infty}(1+e^{-j\frac{\pi k}{4}})X(j(\omega-\frac{\pi k}{T})). \qquad (8)$$

Carefully justify every step in your derivation, including references to results from the tables.

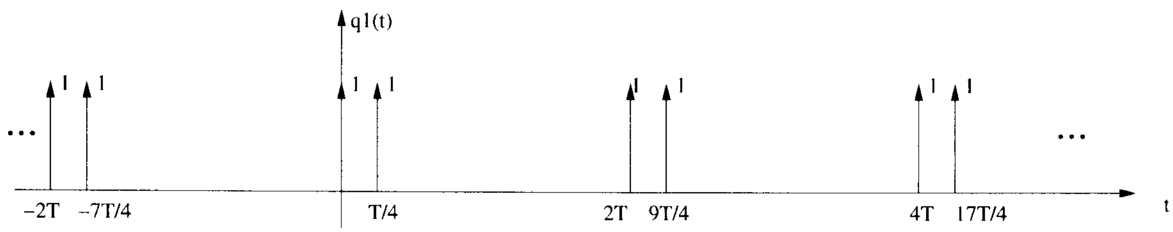


Figure 2: The sampling impulse train $q_1(t)$, where $T = \pi/W$.

$$q(t) = \sum_{n=-\infty}^{\infty} \delta(t - n2T) + \delta(t - n2T - \tfrac{T}{4})$$

$$x(t)q_1(t) \xrightarrow{CTFT} \frac{1}{2\pi} X(j\omega) * Q_1(j\omega)$$

$$Q_1(j\omega) = \frac{2\pi}{2T}\sum_{k=-\infty}^{\infty}\left(\delta(\omega - \tfrac{2\pi}{2T}k) + e^{-\frac{j\omega T}{4}}\delta(\omega - \tfrac{2\pi}{2T}k)\right)$$

$$= \frac{\pi}{T}\sum_{k=-\infty}^{\infty}\left(\delta(\omega - \tfrac{\pi}{T}k) + e^{-\frac{j\omega T}{4}}\delta(\omega - \tfrac{\pi}{T}k)\right)$$

$$= \frac{\pi}{T}\sum_{k=-\infty}^{\infty}\left(\delta(\omega - \tfrac{\pi}{T}k) + e^{-j\frac{\pi k T}{4T}}\delta(\omega - \tfrac{\pi}{T}k)\right)$$

$$Q_1(j\omega) = \frac{\pi}{T}\sum_{k=-\infty}^{\infty}(1 + e^{-\frac{j\pi k}{4}})\,\delta(\omega - \tfrac{\pi}{T}k)$$

Recall:

- $\displaystyle\sum_{n=-\infty}^{\infty}\delta(t - n\Delta) \xrightarrow{CTFT} \frac{2\pi}{\Delta}\sum_{k=-\infty}^{\infty}\delta(\omega - \tfrac{2\pi}{\Delta}k)$

  where $\Delta \in \mathbb{R}_+$

- If $v(t) \xrightarrow{CTFT} V(j\omega)$, then

  $v(t - \tfrac{T}{4}) \xrightarrow{CTFT} e^{-\frac{j\omega T}{4}} V(j\omega)$

- $z(\omega)\delta(\omega - \omega_0) = z(\omega_0)\delta(\omega - \omega_0)$

  so $e^{-\frac{j\omega T}{4}}\delta(\omega - \tfrac{\pi}{T}k) = e^{-\frac{j\pi T}{T4}}\delta(\omega - \tfrac{\pi}{T}k)$

Important to point out.

$$Y_1(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)\, Q_1(j(\omega - \theta))\,d\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)\frac{\pi}{T}\sum_{k=-\infty}^{\infty}(1 + e^{-\frac{j\pi k}{4}})\,\delta(\omega - \theta - \tfrac{\pi}{T}k)\,d\theta$$

$$= \frac{\pi}{2\pi T}\sum_{k=-\infty}^{\infty}(1 + e^{-\frac{j\pi k}{4}})\int_{-\infty}^{\infty} X(j\theta)\,\delta(\omega - \theta - \tfrac{\pi}{T}k)\,d\theta = \frac{1}{2T}\sum_{k=-\infty}^{\infty}(1 + e^{-\frac{j\pi k}{4}})\,X(j(\omega - \tfrac{\pi}{T}k))$$

Sifting sets $\theta = \omega - \frac{\pi}{T}k$.

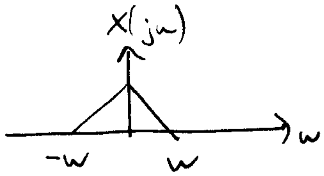
Had to show $e^{-\frac{j\omega T}{4}}$ sifting to $e^{-\frac{j\pi k}{4}}$ either before or during convolution.

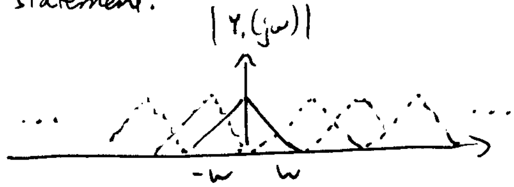
(b) (5 Pts) Is it possible to low-pass filter the signal $y_1(t)$ to get back the signal $x(t)$?

Answer: yes/no.

Explanation: *(Hint: Pick an example band-limited spectrum for $x(t)$, and sketch the resulting spectrum of $y_1(t)$. Based on your plot, explain.)*

There are spectral copies every $\frac{\pi}{T}$. Note that $\frac{\pi}{T} = W$ from the problem statement.





Aliasing occurs. A lowpass filter cannot remove this.

(c) (5 Pts) Suppose the signal is sampled non-uniformly using the impulse train $q_2(t)$ shown in Figure 3. The impulses are in the same locations as in Figure 2, but they have different weights a and b (both real numbers). Find an expression for the spectrum $Y_2(j\omega)$ of the sampled signal $y_2(t) = x(t)q_2(t)$.

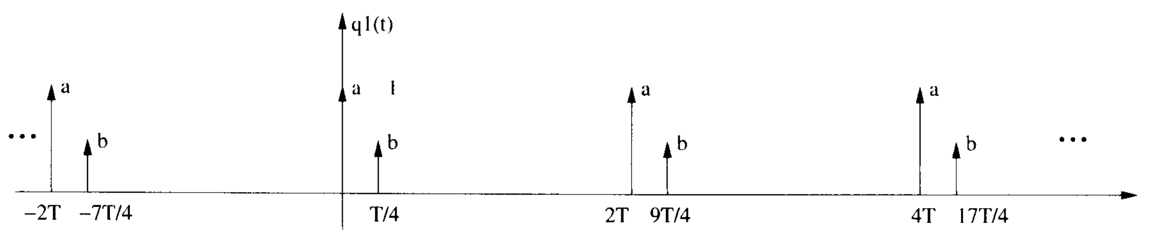


Figure 3: The sampling impulse train $q_2(t)$, where $T = \pi/W$.

By linearity, we can use our own work from part a) to get:

$$\boxed{Y_2(j\omega) = \frac{1}{2T} \sum_{k=-\infty}^{\infty} \left(a + b e^{-j\frac{\pi k}{4}}\right) X\left(j\left(\omega - \frac{\pi}{T}k\right)\right)}$$

$$q_2(t) = \sum_{n=-\infty}^{\infty} a\,\delta(t - n2T) + b\,\delta\left(t - n2T - \frac{T}{4}\right)$$

$$Q_2(j\omega) = \frac{2\pi}{2T} \sum_{k=-\infty}^{\infty} a\,\delta\left(\omega - \frac{2\pi}{2T}k\right) + b e^{-j\frac{\omega T}{4}} \delta\left(\omega - \frac{2\pi}{2T}k\right)$$

Rest follows as before.

(d) (5 Pts) For $|\omega| < W$, write out the spectrum

$$Y(j\omega) = \begin{cases} Y_1(j\omega) + jY_2(j\omega), & \text{if } \omega \geq 0, \\ Y_1(j\omega) - jY_2(j\omega), & \text{if } \omega < 0. \end{cases} \quad (6)$$

Select the real numbers a and b such that for $0 < \omega < W$, it is true that $Y(j\omega) = \frac{1}{2T}(2 + j(a + b))X(j\omega)$. *Hint:* A complex number is zero if and only if its real part is zero *and* its imaginary part is zero.

In $0 < \omega < W$ we have:

$$Y(j\omega) = \frac{1}{2T}\sum_{k=-\infty}^{\infty}\left(1 + e^{-j\frac{\pi k}{4}} + ja + jbe^{-j\frac{\pi k}{4}}\right) X\left(j\left(\omega - \frac{\pi}{T}k\right)\right)$$

Note that only the $k=0$ and $k=1$ copy are in $0 < \omega < W$ because $x(t)$ is bandlimited to W and $W = \frac{\pi}{T}$.

$$Y(j\omega) = \underbrace{\frac{1}{2T}\left(1 + e^{-j\frac{\pi 0}{4}} + ja + jbe^{-j\frac{\pi 0}{4}}\right)}_{\text{Just } 2 + j(a+b)} X(j\omega) + \frac{1}{2T}\underbrace{\left(1 + e^{-j\frac{\pi}{4}} + ja + jbe^{-j\frac{\pi}{4}}\right)}_{\text{Make this zero.}} X\left(j\left(\omega - \frac{\pi}{T}\right)\right)$$

$$1 + \cos\left(\frac{\pi}{4}\right) - j\sin\left(\frac{\pi}{4}\right) + ja + jb\cos\left(\frac{\pi}{4}\right) + b\sin\left(\frac{\pi}{4}\right) = 0$$

Since a and b are real, we can make this two equations.

$$1 + \cos\left(\frac{\pi}{4}\right) + b\sin\left(\frac{\pi}{4}\right) = 0 \Rightarrow b = \frac{-1 - \cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} = \frac{-1 - \frac{\sqrt{2}}{2}}{\sqrt{2}/2} = -1 - \sqrt{2}$$

$$-\cancel{j}\sin\left(\frac{\pi}{4}\right) + \cancel{j}a + \cancel{j}b\cos\left(\frac{\pi}{4}\right) = 0 \Rightarrow a = \sin\left(\frac{\pi}{4}\right) - b\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + (1 + \sqrt{2})\frac{\sqrt{2}}{2} = 1 + \sqrt{2}$$

$$\boxed{a = 1 + \sqrt{2} \qquad b = -1 - \sqrt{2}}$$

(e) (1 Pt) It can also be shown that with the choice of a and b as in Part *(d)*, it is true that $Y(j\omega) = \frac{1}{2T}(2 - j(a + b))X(j\omega)$ for $-W < \omega < 0$. Hence, from $Y(j\omega)$, one can determine $X(j\omega)$ using a simple filter. This means that non-uniform sampling using the sampling intervals shown in Figure 2 (and in Figure 3) permits to perfectly reconstruct $x(t)$ from samples. Give an intuitive explanation why this makes sense.

$\boxed{\omega_s > 2W \text{ on average}}$

We are sampling at the critical rate on average. Since we are sampling non-uniformly, we need to choose our sample heights carefully to avoid aliasing.