**EE 120: Signals and Systems**
Department of EECS
UC BERKELEY

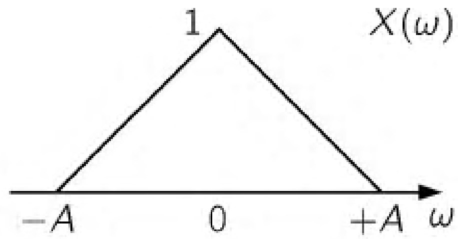
MIDTERM 2
8 April 2009

LAST Name Annihilator FIRST Name Pole-Zero

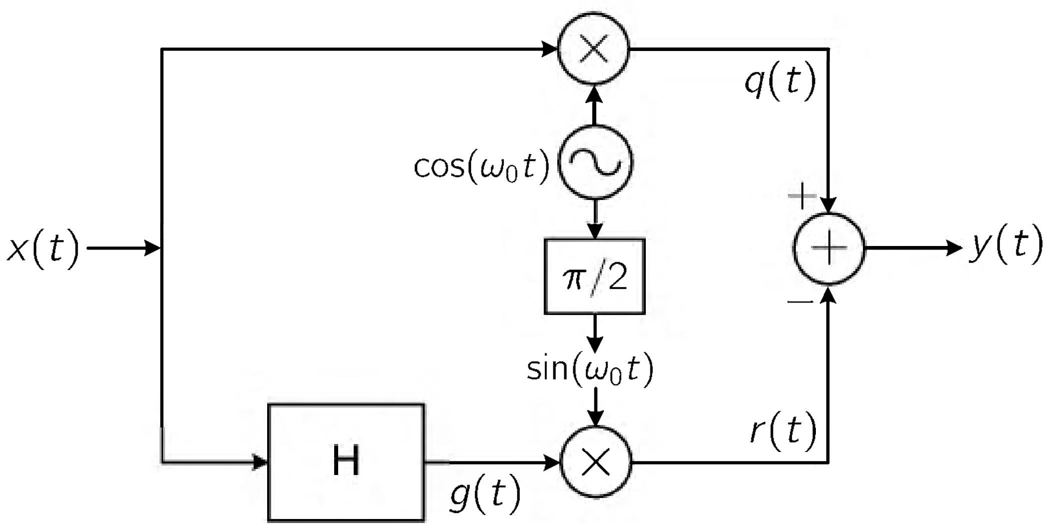
Discussion Time ?

- **(10 Points)** Print your name and discussion time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 90 minutes to complete. You will be given at least 90 minutes, up to a maximum of 110 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except two double-sided $8.5" \times 11"$ sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 10.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the ten numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

**MT2.1 (35 Points)** A bandlimited continuous-time signal $x$ has the spectrum shown below:




The following block diagram depicts an amplitude modulation scheme to transmit the signal $x$:




The frequency response of the LTI system H in the block diagram is given by

$$H(\omega) = -i\,\mathrm{sgn}(\omega), \quad \forall\omega,$$

where

$$\mathrm{sgn}(\omega) = \begin{cases} 1 & \omega \geq 0 \\ -1 & \omega < 0 \end{cases}$$

is the signum function.

Throughout this problem, assume that the carrier frequency is sufficiently large; in particular, let $A \ll \omega_0$.

Please do not forget the negative sign at the adder on the right side of the diagram.

**Facts that may prove useful:**

**Signum Function:**

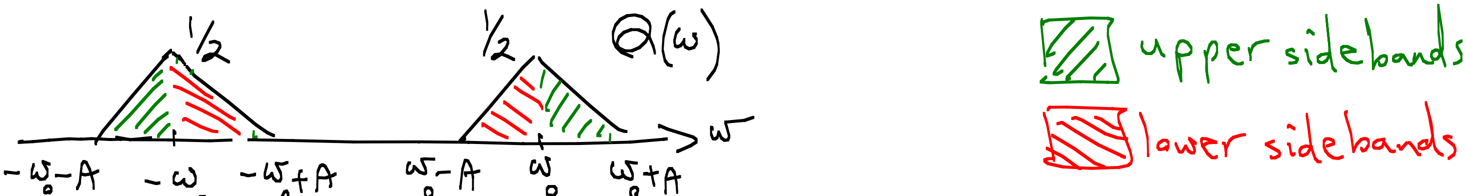
$$\mathcal{F}(\operatorname{sgn}(t)) = \frac{2}{i\omega}.$$

**Duality Property of the CTFT:** If $\mathcal{F}(x(t)) = X(\omega)$, then $\mathcal{F}(X(t)) = 2\pi x(-\omega)$.

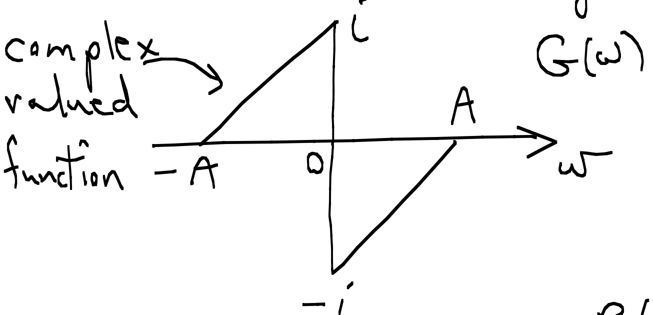
(a) Provide well-labeled plots of $G(\omega)$, $R(\omega)$, $Q(\omega)$, and $Y(\omega)$, the CTFTs of the signals $g$, $r$, $q$, and $y$, respectively.

Be sure to explain your reasoning succinctly, but clearly and convincingly.

$$q(t) = x(t)\cos(\omega_0 t) \Rightarrow Q(\omega) = \frac{1}{2}\left[X(\omega+\omega_0) + X(\omega-\omega_0)\right]$$

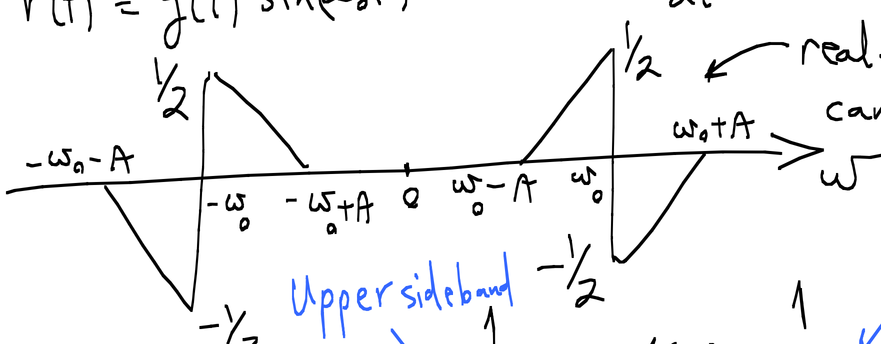



$$G(\omega) = X(\omega)H(\omega) = -i\operatorname{sgn}\omega\, X(\omega) = \begin{cases} -i\,X(\omega) & \omega > 0 \\ +i\,X(\omega) & \omega < 0 \end{cases}$$



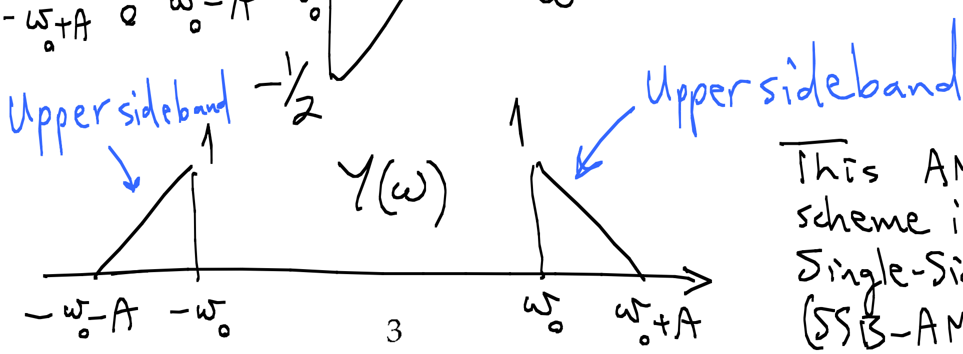

This is neither a magnitude nor a phase plot.

$$r(t) = g(t)\sin(\omega_0 t) \Rightarrow R(\omega) = \frac{1}{2i}\left[G(\omega-\omega_0) - G(\omega+\omega_0)\right]$$




real-valued function because of cancelation of $i$.

$$Y(\omega) = Q(\omega) - R(\omega)$$




This AM transmission scheme is called Single-Sideband AM (SSB-AM)

(b) Determine a simple expression for, and provide a well-labeled plot of, $h(t)$, the impulse response of the filter H.

$H(\omega) = -i \operatorname{sgn} \omega$

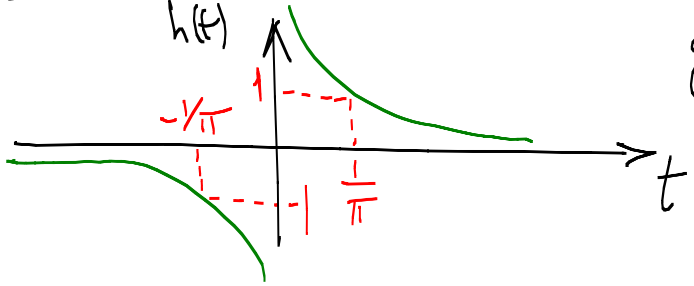
We know $\operatorname{sgn}(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{2}{i\omega}$. Using the duality property of the CTFT, we know

$$\frac{2}{it} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi \operatorname{sgn}(-\omega) = -2\pi \operatorname{sgn} \omega \quad (\text{noting that } \operatorname{sgn}(-\omega) = -\operatorname{sgn}\omega)$$

With simple scaling in the time & frequence domains we have:

$$\frac{1}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} -i \operatorname{sgn}(\omega) \implies h(t) = \frac{1}{\pi t}$$

The filter h is called a Hilbert Transformer, and



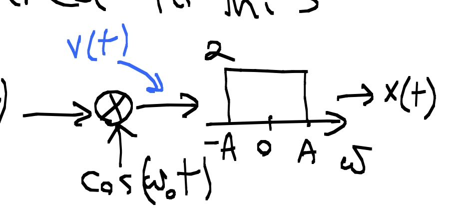

$$g(t) = (x * h)(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t - \tau} \, d\tau$$

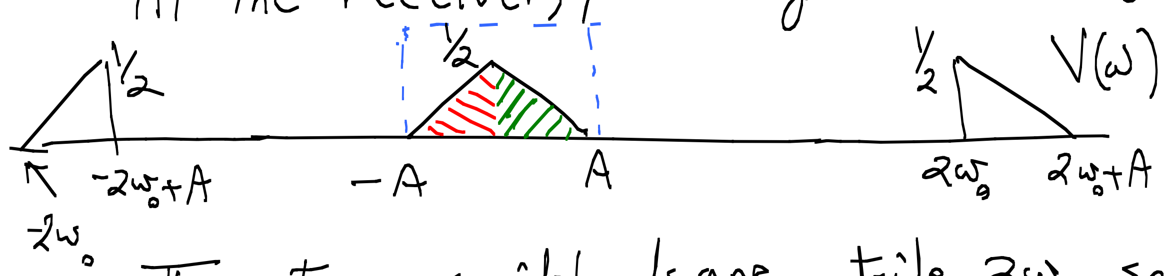
is called the Hilbert transform of x.

(c) Explain why transmitting the signal $y$ might be preferable to transmitting the signal $q$ alone. Be succinct, but clear and convincing.

This scheme saves valuable spectrum. Compare how much real-estate space along the spectrum the ordinary AM scheme $q(t) = x(t)\cos(\omega_0 t)$ occupies compared to this scheme.

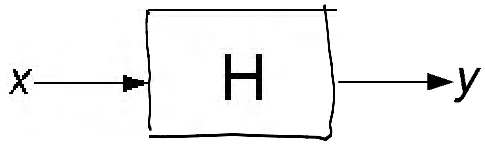
At the receiver, process $y(t)$ as follows:




The extraneous sidebands are outside $2\omega_0$, so we have even more leeway in our choice of low-pass filter: don't need as sharp a cutoff.

**MT2.2 (35 Points)** A discrete-time, LTI, causal, FIR filter H is shown below:




We have the following information about this filter; synthesize each hint in the space provided to receive partial credit.

- The impulse response of the filter is real-valued: $h(n) \in \mathbb{R}, \forall n$. (What are implications of this for the poles and zeros of the filter's transfer function?)

$$h(n) \in \mathbb{R} \implies \hat{H}^*(z) = \sum_n h^*(n)(z^{-n})^* = \sum_n h(n)(z^*)^{-n} = H(z^*) \implies$$

If $z = z_0$ is a zero / pole of $\hat{H}$, then so is $z_0^*$ a zero / pole $\hat{H} \implies$

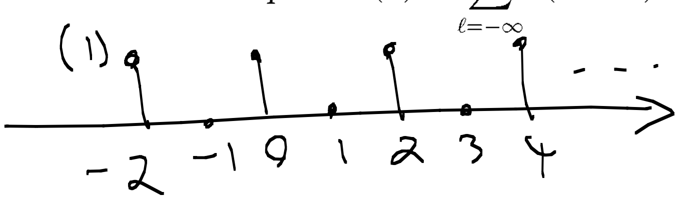
Each pole (zero) of $H$ is either real or appears with its complex conjugate.

- If the input is $x(n) = \cos\left(\frac{\pi}{2}n\right), \forall n$, then the output is $y(n) = 0, \forall n$.

$$x(n) = \cos\left(\frac{\pi}{2}n\right) = \frac{1}{2}e^{i\pi/2\, n} + \frac{1}{2}e^{-i\pi/2\, n} \implies y(n) = \frac{1}{2}\hat{H}(i)e^{i\pi/2\, n} + \frac{1}{2}\hat{H}(-i)e^{-i\pi/2\, n} \implies$$

$\hat{H}(i) = \hat{H}(-i) = 0 \implies i$ & $-i$ are zeros of $\hat{H}(z)$. Since H is FIR, $\hat{H}(z)$ must be rational in $z \implies$ Two factors in $\hat{H}(z)$ are $(z-i)$ & $(z+i)$, or simply $(z-i)(z+i) = z^2+1$

- If the input is $x(n) = \sum_{\ell=-\infty}^{\infty} \delta(n - 2\ell), \forall n$, the output is $y(n) = 0, \forall n$.




$x(n)$

$$p = 2 \implies \omega_0 = \frac{2\pi}{2} = \pi$$

$$x(n) = \frac{1}{2}\sum_{n=0}^{1} x(n)e^{ik\pi n}$$

$$x(n) = \frac{1}{2}\left[1 + (-1)^n\right] = \frac{1}{2}\left(1 + e^{i\pi n}\right) \implies y(n) = \frac{1}{2}\hat{H}(1) + \frac{1}{2}\hat{H}(-1)e^{i\pi n} = 0 \quad \forall n$$

$\implies \hat{H}(1) = \hat{H}(-1) = 0 \implies \hat{H}(z)$ has zeros at $z = 1$ & $z = -1$

corresponds to freq 0    freq $\pi$

$\implies (z+1)(z-1) = z^2 - 1$ is another factor in $\hat{H}(z)$.

- The transfer function $\widehat{H}$ of the filter satisfies the following equality:

$$\widehat{H}(1/z) = -z^4 \widehat{H}(z).$$

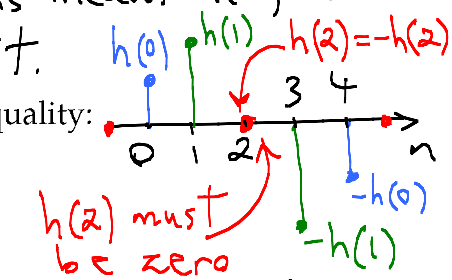
  What's the implication of this on the impulse response?

$$\hat{H}\left(\tfrac{1}{z}\right) = \sum_n h(n)\left(1/z\right)^{-n} = \sum_n h(n)\, z^{n} = \sum_m h(-m)\, z^{-m}, \text{ where } m=-n$$

$$-z^4 H(z) = -z^4 \sum_n h(n)\, z^{-n} = -\sum_n h(n)\, z^{-(n-4)} = -\sum_m h(m+4)\, z^{-m} \quad m = n-4$$

So we must have $h(-m) = -h(m+4) \xrightarrow{n=-m} \underline{h(n) = -h(4-n)}$ (☆)

H is causal $\Rightarrow h(n)=0$ $n<0$. Coupled w/ (☆) this means $h(n)=0$ outside of the interval $[0,4]$ and is anti-symmetric inside it.




$$\Downarrow$$

$$\hat{H}(z) = \frac{K(z^2+1)(z^2-1)}{z^4}$$

- The frequency response of the filter satisfies the following equality:

$$\int_{\langle 2\pi \rangle} H(\omega)\, e^{i4\omega} d\omega = -2\pi.$$

$$h(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} H(\omega)\, e^{i\omega n} d\omega \Rightarrow h(4) = \frac{1}{2\pi} \underbrace{\int_{\langle 2\pi\rangle} H(\omega)\, e^{i4\omega} d\omega}_{-2\pi}$$

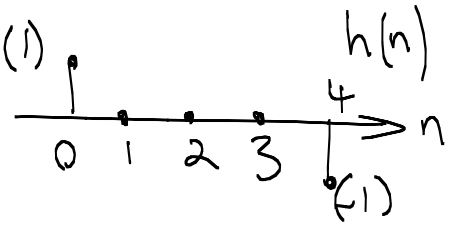
So $h(4) = \frac{-2\pi}{2\pi} = -1 \Rightarrow h(0) = +1$

Based on your synthesis of the hints above, answer each of the following questions:

(a) Determine and provide a well-labeled plot of the impulse response $h$ of the filter.

$$\hat{H}(z) = \frac{K(z^2-1)(z^2+1)}{z^4} = K\,\frac{z^4-1}{z^4} = K\left(1 - z^{-4}\right) \Rightarrow$$

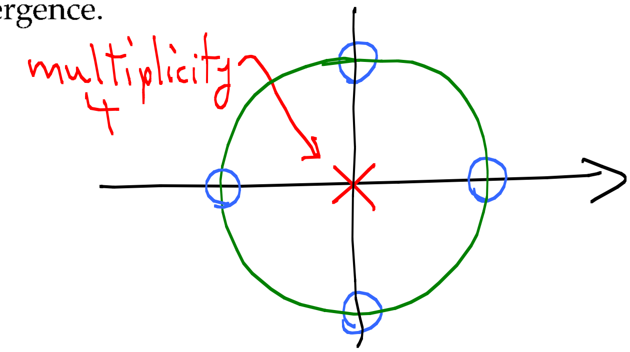
$h(1) = h(3) = 0$. We must have $K=1$ to make $h(0) = 1 = -h(4)$



(b) Determine a simple expression for $\widehat{H}(z)$, the transfer function of the filter. Specify the corresponding region of convergence.

$$\boxed{\hat{H}(z) = 1 - z^{-4}}$$

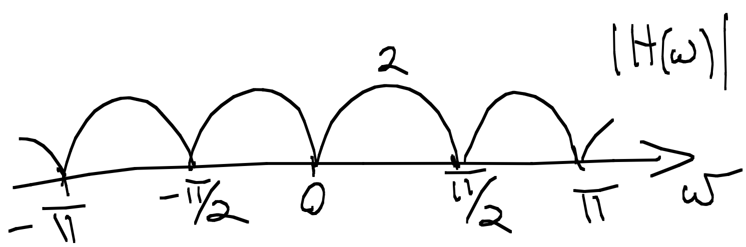
$$R_h: \quad 0 < |z|$$




(c) Provide well-labeled plots of $|H(\omega)|$ and $\angle H(\omega)$, the magnitude and phase response of the filter.

$$H(\omega) = \hat{H}(z)\Big|_{z=e^{i\omega}} \Rightarrow H(\omega) = 1 - e^{-i4\omega} = \left(e^{i2\omega} - e^{-i2\omega}\right)e^{-i2\omega} \Rightarrow$$

$$\boxed{H(\omega) = 2i\sin(2\omega)e^{-i2\omega}} \Rightarrow |H(\omega)| = 2|\sin(2\omega)|$$




$$\boxed{H(\omega) = 2\sin(2\omega)e^{i\left(\frac{\pi}{2} - 2\omega\right)}}$$

$\sin(2\omega) > 0$

$$\angle H(\omega) = \begin{cases} \frac{\pi}{2} - 2\omega & |\omega| < \frac{\pi}{2} \\ +\frac{3\pi}{2} - 2\omega & \frac{\pi}{2} < |\omega| < \pi \end{cases}$$

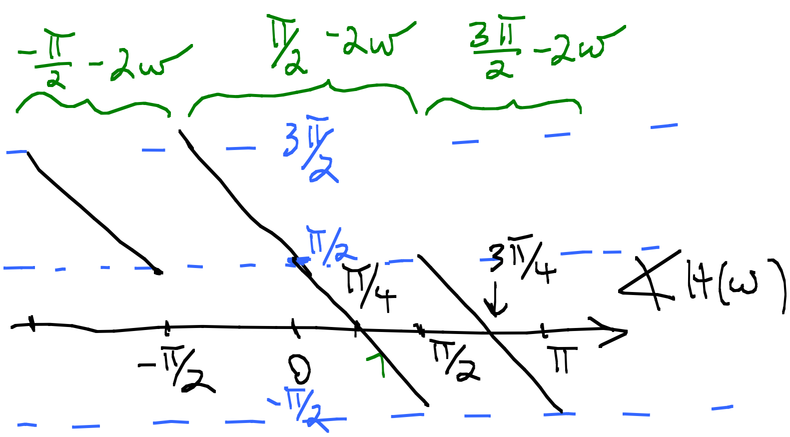
$\sin(2\omega) < 0$

(contributes phase of $\pm\pi$)

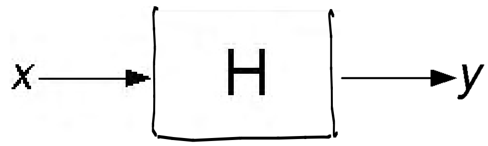
alternatively,

$$\frac{3\pi}{2} - 2\omega - 2\pi = -\frac{\pi}{2} - 2\omega$$

can add or subtract any integer multiple of $2\pi$ to or from phase



**MT2.3 (35 Points)** A discrete-time LTI filter H is shown below:




The transfer function of the filter is given by the following expression:

$$\widehat{H}(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n^2}.$$

In one or more parts below, you may find the following identity helpful:

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^2} = \frac{\theta^2}{4} - \frac{\pi\,\theta}{2} + \frac{\pi^2}{6}, \quad \text{for } 0 \leq \theta \leq 2\pi.$$

(a) Determine a simple expression for, and provide a well-labeled plot of, the filter's impulse response $h(n)$.

$$\hat{H}(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n^2} = \sum_{n=-\infty}^{\infty} h(n) z^{-n} \Rightarrow h(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{n^2} & n \geq 1 \end{cases} \Rightarrow h(n) = \frac{1}{n^2} u(n-1)$$

(b) Select the strongest true assertion from the list below.

$\boxed{\text{(i) The filter must be causal.}}$

(ii) The filter could be causal, but does not have to be.

(iii) The filter cannot be causal.

Explain your reasoning succinctly, but clearly and convincingly.

$h(n) = 0 \quad \forall n < 0$

(c) Select the strongest true assertion from the list below.

(i) $\boxed{\text{The filter must be BIBO stable.}}$

(ii) The filter could be BIBO stable, but does not have to be.

(iii) The filter cannot be BIBO stable.

Explain your reasoning succinctly, but clearly and convincingly.

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

(d) Determine a reasonably simple expression for, and provide a well-labeled plot of, the output $y(n)$ of the filter if the input is given by

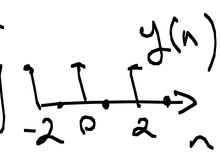
$$x(n) = \frac{3}{\pi^2}\left[1 - 2(-1)^n\right].$$

$$x(n) = \frac{3}{\pi^2}\left[1^n - 2(-1)^n\right] \Rightarrow y(n) = \frac{3}{\pi^2}\left[\hat{H}(1) - 2\hat{H}(-1)(-1)^n\right]$$

$$\hat{H}(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{\cos(n\cdot 0)}{n^2} = \pi^2/6$$

$$\hat{H}(-1) = \sum_{n=1}^{\infty} \frac{(-1)^{-n}}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2} = \frac{\pi^2}{4} - \frac{\pi^2}{2} + \frac{\pi^2}{6} = -\pi^2/12$$

$$y(n) = \frac{3}{\pi^2}\left[\pi^2/6 - 2(-\pi^2/12)(-1)^n\right] = \frac{3}{\cancel{\pi^2}} \cdot \frac{\cancel{\pi^2}}{6}\left[1 + (-1)^n\right] = \frac{1}{2}\left[1 + (-1)^n\right]$$




(e) Without attempting to determine the filter's frequency response $H(\omega)$, find a numerical value for

$$\int_{\langle 2\pi \rangle} H(\omega)\, d\omega$$

$$h(n) = \frac{1}{2\pi}\int_{\langle 2\pi \rangle} H(\omega) e^{i\omega n}\, d\omega \Rightarrow h(0) = \frac{1}{2\pi}\int_{\langle 2\pi \rangle} H(\omega)\, d\omega \Rightarrow \int_{\langle 2\pi \rangle} H(\omega)\, d\omega = 2\pi h(0)$$

We know $h(0) = 0 \Rightarrow \int_{\langle 2\pi \rangle} H(\omega)\, d\omega = 0$

Also note: $h(0) = \lim_{z\to\infty} \hat{H}(z) = 0$ for this causal filter

LAST Name Annihilator FIRST Name Pole-Zero

Discussion Time ?

| Problem Name | Points 10 | Your Score 10 |
|---|---|---|
| 1 | 35 | 35 |
| 2 | 35 | 35 |
| 3 | 35 | 35 |
| **Total** | **115** | 115 |