MT1.1 (35 Points) The frequency response of a causal discrete-time LTI filter H is

$$\forall \omega \in \mathbb{R}, \quad H(\omega) = A\frac{1-e^{-i2\omega}}{1+R^2 e^{-i2\omega}},$$

where $R^2 = 0.96$ and $A = 1/50$.

(a) Determine the linear, constant-coefficient difference equation that characterizes the input-output behavior of the system.

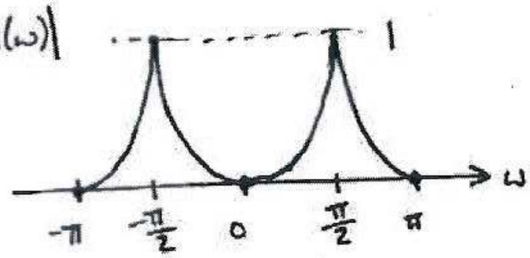
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = A\frac{1-e^{-i2\omega}}{1+R^2 e^{-i2\omega}} \Rightarrow Y(\omega)(1+R^2 e^{-i2\omega}) = X(\omega)A(1-e^{-i2\omega})$$

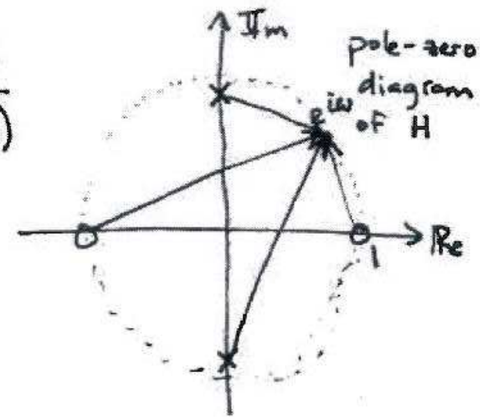
Using time-shift prop: $y(n)+R^2 y(n-2) = [x(n)-x(n-2)]A$

(b) Provide a well-labeled plot of $|H(\omega)|$, the magnitude response of the filter. You must explain how you arrive at the plot.

$$H(\omega) = A\frac{e^{i2\omega}-1}{e^{i2\omega}+R^2} = A\frac{(e^{i\omega}-1)(e^{i\omega}+1)}{(e^{i\omega}-iR)(e^{i\omega}+iR)}$$

$|H(\omega)|$





$H(0) = H(\pi) = H(-\pi) = 0$ because of zeros

$$H(\tfrac{\pi}{2}) = H(-\tfrac{\pi}{2}) = A\frac{1-(-1)}{1-R^2} = 1$$

$$\left|H(\tfrac{\pi}{4})\right| = \left|A\frac{1+i}{1-iR^2}\right| \approx A$$

(c) Determine the output of the filter in response to the input signal

$$\forall n \in \mathbb{Z}, \quad x(n) = 1 + (-1)^n + \cos\left(\frac{\pi}{2}n\right).$$

$$x(n) = \underbrace{e^{i\cdot 0\cdot n} + e^{i\cdot\pi\cdot n}}_{\text{will be filtered since } H(0)=H(\pi)=0} + \underbrace{\tfrac{1}{2}e^{i\frac{\pi}{2}n} + \tfrac{1}{2}e^{-i\frac{\pi}{2}n}}_{\text{will remain intact since } H(\frac{\pi}{2})=H(-\frac{\pi}{2})=1}$$

Output $y(n) = \cos(\frac{\pi}{2}n)$

MT1.2 (35 Points) A continuous-time signal x is periodic with fundamental period $p = 6$ seconds. We sample this signal every $T = 3$ seconds to produce a discrete-time signal g as follows:

$$\forall n \in \mathbb{Z}, \quad g(n) = x(nT).$$

(a) Show that $g(n+2) = g(n)$, for all n.

$$g(n) = x(3n)$$

$$g(n+2) = x(3(n+2)) = x(3n+6) \overset{\text{since } x(t+6)=x(t)}{=} x(3n) = g(n)$$

(b) Express the DFS coefficients G_ℓ of the DT signal g in terms of the CTFS coefficients X_k of the CT signal x.

x: periodic with period $p = 6$ sec. $\rightarrow \omega_{0x} = \frac{2\pi}{6} = \frac{\pi}{3}$

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_{0x}t} = \sum_{k=-\infty}^{\infty} X_k e^{ik\frac{\pi}{3}t} \Rightarrow g(n) = x(3n) = \sum_{k=-\infty}^{\infty} X_k e^{ik\pi n}$$

$$= \sum_{k=-\infty}^{\infty} X_k (-1)^{kn}$$

g: periodic with period $q = 2$ samples $\rightarrow \omega_{0g} = \frac{2\pi}{2} = \pi$

$$G_\ell = \frac{1}{q} \sum_{n=\langle q \rangle} g(n) e^{-i\ell\omega_{0g}n} = \frac{1}{2} \sum_{n=0}^{1} g(n) e^{-i\ell\pi n}$$

$$= \frac{1}{2} \sum_{n=0}^{1} \left(\sum_{k=-\infty}^{\infty} X_k (-1)^{kn} \right) (-1)^{\ell n} = \frac{1}{2} \sum_{n=0}^{1} \sum_{k=-\infty}^{\infty} X_k (-1)^{(k+\ell)n}$$

$$= \frac{1}{2} \left(\sum_{k=-\infty}^{\infty} X_k + \sum_{k=-\infty}^{\infty} X_k (-1)^{k+\ell} \right) = \underline{\frac{1}{2} \sum_{k=-\infty}^{\infty} X_k \left(1 + (-1)^{k+\ell}\right) = G_\ell}$$

for $\ell = 0, 1$

$$G_0 = \frac{1}{2} \sum_{k=-\infty}^{\infty} X_k \underbrace{\left(1 + (-1)^k\right)}_{= \begin{cases} 2, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}} = \sum_{k \text{ even}} X_k$$

$$G_1 = \frac{1}{2} \sum_{k=-\infty}^{\infty} X_k \underbrace{\left(1 + (-1)^{k+1}\right)}_{= \begin{cases} 0, & k \text{ even} \\ 2, & k \text{ odd} \end{cases}} \overset{3}{=} \sum_{k \text{ odd}} X_k$$

MT1.3 (35 Points) The spectrum of a periodic DT signal x is given by

$$\forall \omega \in \mathbb{R}, \quad X(\omega) = \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{\pi}{3} + 2\pi k\right) + \delta\left(\omega - \frac{3\pi}{5} + 2\pi k\right),$$

(a) Determine a reasonably simple expression for $x(n)$, for all n.

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{i\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\delta(\omega - \tfrac{\pi}{3}) + \delta(\omega - \tfrac{3\pi}{5})\right) e^{i\omega n} d\omega$$

$$= \frac{1}{2\pi}\left(e^{i\frac{\pi}{3}n} + e^{i\frac{3\pi}{5}n}\right) = \frac{1}{2\pi} e^{i\frac{7\pi}{15}n}\left(e^{-i\frac{2\pi}{15}n} + e^{i\frac{2\pi}{15}n}\right)$$

$$x(n) = \frac{1}{\pi} e^{i\frac{7\pi}{15}n} \cos\left(\frac{2\pi}{15}n\right)$$

(b) Determine the fundamental period, the fundamental frequency, and the DFS coefficients of x. How many of the coefficients are zero?

Fundamental period $p = 30$ (Check that $x(n+30) = x(n)$)

Fundamental frequency $\omega_0 = \frac{2\pi}{p} = \frac{\pi}{15}$

$$x(n) = \sum_{k=\langle p\rangle} X_k e^{ik\omega_0 n} = \sum_{k=\langle 30\rangle} X_k e^{ik\frac{\pi}{15}n} = \frac{1}{2\pi} e^{i5\frac{\pi}{15}n} + \frac{1}{2\pi} e^{i9\frac{\pi}{15}n}$$

$\Rightarrow X_5 = X_9 = \frac{1}{2\pi}$ & The other 28 coefficients are 0

1.2 (c) Explain whether g is guaranteed to be periodic if p and T are arbitrary positive real numbers.

No, g is not guaranteed to be periodic unless $np = kT$ for some positive integers n and k (i.e., $\frac{p}{T}$ must be rational)

Counterexample: $p=\pi$, $T=2$.

$x(t) = x(t+\pi)$

$x(2n) = x(nT) = x(nT+\pi) = x\left(\left(n+\frac{\pi}{T}\right)T\right) = x\left(\left(n+\frac{\pi}{2}\right)2\right) = g\left(n+\frac{\pi}{2}\right)$

but this doesn't exist since $\frac{\pi}{2}$ is not an integer

g will not be periodic