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EE 121 - Midterm Solutions
Spring 1997

Problem 1

a) False. If X and Y are continuous-valued random variables, then

$$\begin{aligned} E(X+Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)f_{X,Y}(x,y)dxdy \\ &= \int_{-\infty}^{\infty} x\int_{-\infty}^{\infty} f_{X,Y}(x,y)dydx + \int_{-\infty}^{\infty} y\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy \\ &= \int_{-\infty}^{\infty} xf_X(x)dx + \int_{-\infty}^{\infty} yf_Y(y)dy \\ &= E(X)+E(Y) \end{aligned}$$

Hence, expectation is linear and do not depend on the independence of X and Y.

b) True.

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$

If X and Y are uncorrelated, then $Cov(X,Y) = 0$.

c) False. This is only true for **independent** discrete-valued random variables.

d) False.

$$m_Y(t) = E(Y(t)) = E(Xt) = t * m_X$$

Clearly, the mean of $\{Y(t)\}$ is dependent on time. Therefore, $\{Y(t)\}$ is **not** strict sense stationary.

e) False. Consider this counterexample. Let's say our experiment is two fair coin tosses. Let X be 1 if the first toss is heads, 0 otherwise. Let Y be 1 if the second toss is heads, 0 otherwise. Let Z be 1 if the two tosses are the same, 0 if they are different. Clearly, X, Y, and Z are pairwise independent.

If X, Y, and Z are all independent, then $P(Z=1|X=1,Y=1)$ should equal $P(Z=1)$. $P(Z=1|X=1,Y=1)$ is easily seen to be 1, but $P(Z=1) = 1/2$. Since the two are not equal, X, Y, and Z are not independent.

Problem 2

a) $\mathcal{S}_W(f) = N_0/2$. Therefore,

$$E[|W(t)|^2] = R_W(\tau)|_{\tau=0} = \frac{N_0}{2}\delta(0) = \infty$$

b) Since $H(f)$ is a stable LTI system and the input $\{W(t)\}$ is a WSS process, the output $\{Y(t)\}$ is also WSS. Furthermore,

$$\mathcal{S}_Y(f) = |H(f)|^2\mathcal{S}_W(f) = |H(f)|^2\frac{N_0}{2}$$

c)

$$\mathcal{S}_Y(f) = |H(f)|^2\frac{N_0}{2} = \frac{N_0}{2}\Pi(\frac{f}{2W})$$

Taking the inverse Fourier transform,

$$R_Y(\tau) = N_0W\text{sinc}(2W\tau)$$

Then,

$$E[|Y(t)|^2] = R_Y(\tau)|_{\tau=0} = N_0 W \text{sinc}(0) = N_0 W$$

d) Sampling the output $\{Y(t)\}$ at rate $1/T = 2W$ samples per second means that we are looking at samples $1/2W$ seconds apart. The time difference between any two samples of this discrete-time process $\{Y(nT)\} = \{Y[n]\}$ is $k/2W$ where k is an integer. Therefore,

$$R_{Y[n]}(k) = R_Y(k/2W) = N_0 W \text{sinc}(k) = N_0 W \delta[k]$$

The process is white because its power-spectral density is flat. Furthermore, it is Gaussian since the samples are from a continuous-time Gaussian process.

e) Now we are looking at samples $1/4W$ seconds apart. The time difference between any two samples is $k/4W$ where k is an integer. Therefore,

$$R_{Y[n]}(k) = R_Y(k/4W) = N_0 W \text{sinc}(\frac{k}{2})$$

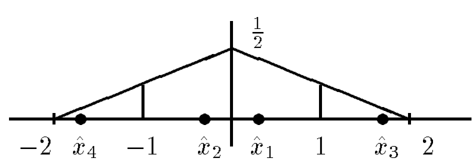
f) Finding the optimal linear predictor consists of solving the Yule-Walker equations for the set of predictor coefficients $\{a_i\}|_{i=1}^{p}$. Since we are only predicting the current sample from the previous sample, p is 1, which means that there is only one coefficient a to solve for.

$$\begin{aligned} aR_{Y[n]}(1-1) &= R_{Y[n]}(1) \\ a &= \frac{R_{Y[n]}(1)}{R_{Y[n]}(0)} \end{aligned}$$

At rate $2W$, $a = \frac{N_0 W \delta[1]}{N_0 W \delta[0]} = 0$. At rate $4W$, $a = \frac{N_0 W \text{sinc}(1/2)}{N_0 W \text{sinc}(0)} = \text{sinc}(1/2) = 2/\pi$. It is evident that using DPCM for the $4W$ case is more beneficial than using PCM because consecutive samples are correlated. However, using DPCM for the $2W$ case provides no benefits because consecutive samples are uncorrelated. In this case, we cannot predict the value of the current sample from the value of the previous sample.

Problem 3

a) Here's the picture of $f_X(x)$, the quantization levels, and the quantization regions.



$$\text{SQNR} = \frac{E(X^2)}{E[(X - Q(X))^2]}$$

The signal power is

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2 f_X(x)dx = 2\int_0^2 x^2 \frac{1}{4}(2-x)dx \\ &= \frac{1}{2}\left(\frac{2x^3}{3} - \frac{x^4}{4}\right)\Big|_0^2 = \frac{1}{2}\left(\frac{16}{3} - \frac{16}{4}\right) \\ &= \frac{2}{3} \end{aligned}$$

The quantization noise power is

$$
\begin{aligned}
E[(X-Q(X))^2] &= \int_{-\infty}^{\infty} (x-Q(x))^2 f_X(x)dx \\
&= 2\int_0^1 \left(x-\frac{1}{3}\right)^2 \frac{1}{4}(2-x)dx + 2\int_1^2 \left(x-\frac{5}{3}\right)^2 \frac{1}{4}(2-x)dx \\
&= \frac{1}{2}\left(\int_0^1 \left(x^2-\frac{2}{3}x+\frac{1}{9}\right)(2-x)dx + \int_1^2 \left(x^2-\frac{10}{3}x+\frac{25}{9}\right)(2-x)dx\right) \\
&= \frac{1}{2}\left(\int_0^1 \left(-x^3+\frac{8}{3}x^2-\frac{13}{9}x+\frac{2}{9}\right)dx + \int_1^2 \left(-x^3+\frac{16}{3}x^2-\frac{85}{9}x+\frac{50}{9}\right)dx\right) \\
&= \frac{1}{2}\left(-\frac{1}{4}x^4+\frac{8}{9}x^3-\frac{13}{18}x^2+\frac{2}{9}x\right)\Bigg|_0^1 + \frac{1}{2}\left(-\frac{1}{4}x^4+\frac{16}{9}x^3-\frac{85}{18}x^2+\frac{50}{9}x\right)\Bigg|_1^2 \\
&= \frac{1}{2}\left(\frac{5}{36}\right)+\frac{1}{2}\left(\frac{3}{36}\right) \\
&= \frac{1}{9}
\end{aligned}
$$

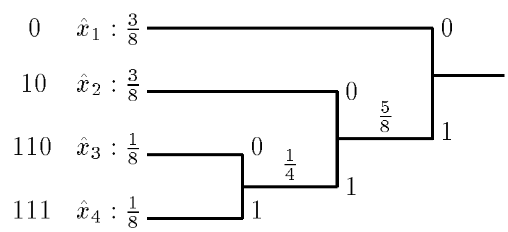
Therefore,

$$\text{SQNR} = \frac{2/3}{1/9} = 6$$

b) The optimal lossless coder to minimize the average bit rate of the coded stream, assuming that we are coding sample-by-sample, is a Huffman coder. To find the code, we need to calculate the probabilities of each quantized value.

$$
\begin{aligned}
P(Q(X_n)=\hat{x}_1) &= P(Q(X_n)=\hat{x}_2) = \frac{1}{4}*1+\frac{1}{2}*1*\frac{1}{4} = \frac{3}{8} \\
P(Q(X_n)=\hat{x}_3) &= P(Q(X_n)=\hat{x}_4) = \frac{1}{2}*1*\frac{1}{4} = \frac{1}{8}
\end{aligned}
$$

Here's a possible Huffman tree.



The minimum average bit rate is

$$
\begin{aligned}
\bar{R}(Q(X_n)) &= \frac{3}{8}*(1)+\frac{3}{8}*(2)+\frac{1}{8}*(3)+\frac{1}{8}*(3) = \frac{15}{8} \\
&= 1.875
\end{aligned}
$$

Problem 4
a) Since $s_4(t) = -s_3(t)$, the dimension of the signal space is 3. We will use the Gram-Schmidt procedure on $\{s_i(t)\}|_{i=1}^3$ to construct an orthonormal basis for the set.

The energy of $s_1(t)$ is

$$E_{s_1} = \int_0^T \sin^2\frac{2\pi t}{T}dt = \int_0^T \left(\frac{1-\cos\frac{4\pi t}{T}}{2}\right)dt = \frac{1}{2}T$$

So the first basis signal is

$$\psi_1(t) = \frac{s_1(t)}{\sqrt{E_{s_1}}} = \sqrt{\frac{2}{T}}s_1(t)$$

$s_1(t)$ and $s_2(t)$ are orthogonal ($s_1(t) \cdot s_2(t) = 0$), hence

$$\psi_2(t) = \frac{s_2(t)}{\sqrt{E_{s_2}}} = \sqrt{\frac{2}{T}}s_2(t)$$

The component of the projection of $s_3(t)$ on $\psi_2(t)$ is 0, but for the projection of $s_3(t)$ on $\psi_1(t)$,

$$\begin{aligned} s_3(t) \cdot \psi_1(t) &= \int_0^{T/2} \sqrt{\frac{2}{T}} \sin\frac{2\pi t}{T} dt = \sqrt{\frac{2}{T}}\frac{T}{\pi} \\ &= \frac{\sqrt{2T}}{\pi} \end{aligned}$$

So, a signal orthogonal to $\psi_1(t)$ and $\psi_2(t)$ is

$$d_3(t) = s_3(t) - \frac{\sqrt{2T}}{\pi}\psi_1(t)$$

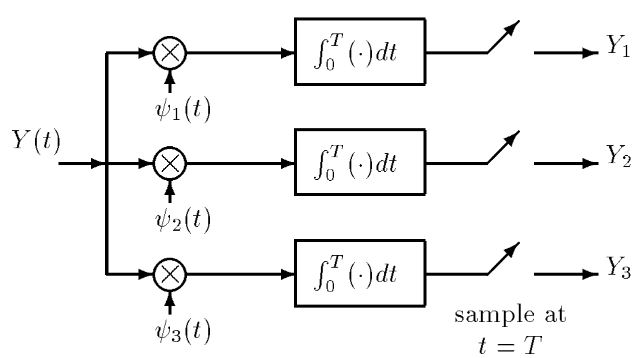
Its energy is

$$\begin{aligned} E_{d_3} &= [s_3(t) - \frac{\sqrt{2T}}{\pi}\psi_1(t)] \cdot [s_3(t) - \frac{\sqrt{2T}}{\pi}\psi_1(t)] \\ &= (s_3(t) \cdot s_3(t)) - \frac{2\sqrt{2T}}{\pi}(s_3(t) \cdot \psi_1(t)) + \frac{2T}{\pi^2}(\psi_1(t) \cdot \psi_1(t)) \\ &= \frac{T}{2} - \frac{2\sqrt{2T}}{\pi}\frac{\sqrt{2T}}{\pi} + \frac{2T}{\pi^2} \\ &= \frac{T}{2} - \frac{2T}{\pi^2} \end{aligned}$$

Then the third basis signal is

$$\psi_3(t) = \left(\frac{T}{2} - \frac{2T}{\pi^2}\right)^{-1/2} \left(s_3(t) - \frac{\sqrt{2T}}{\pi}\psi_1(t)\right)$$

The decorrelator looks like this



$\{Y_i = \int_0^T Y(t)\psi_i(t)dt\}|_{i=1}^3$ are sufficient statistics for optimal detection. Three sufficient statistics are needed because there are three basis signals for the signal set.

b) Yes, the four components do provide sufficient information for optimal detection. The basis signals $\{\psi_i(t)\}|_{i=1}^{3}$ were calculated from $\{s_k(t)\}|_{i=1}^{4}$, hence we can find $\{Y_i = Y(t) \cdot \psi_i(t)\}$ from $\{Y(t) \cdot s_k(t)\}$. Since the $\{Y_i\}$ are sufficient statistics for optimal detection, it follows that $\{Y(t) \cdot s_k(t)\}$ is as well.