Midterm Solutions

1.

$$\begin{aligned} m(t) &= 3\Pi(t/4) - 2\Pi(t/2) \\ \mathcal{F}(m(t)) &= 12\text{sinc}(4f) - 4\text{sinc}(2f). \end{aligned}$$

2. (a) Calculating the Fourier series of $m(t)$ we find that

$$\begin{aligned} m(t) &= \sum_{n=-\infty}^{\infty} \Pi\left(\frac{t-2nT}{T}\right) \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{2}\text{sinc}\left(\frac{n}{2}\right) e^{jn\pi t/T} \\ &= 1/2 + \sum_{n=1}^{\infty} \text{sinc}\left(\frac{n}{2}\right) \cos(n\pi t/T). \end{aligned}$$

Then we have

$$\hat{m}(t) = \sum_{n=1}^{\infty} \text{sinc}\left(\frac{n}{2}\right) \sin(n\pi t/T).$$

(b)

$$\begin{aligned} s(t) &= m(t)\cos(2\pi f_c t) - \hat{m}(t)\sin(2\pi f_c t) \\ &= \left(1/2 + \sum_{n=1}^{\infty} \text{sinc}\left(\frac{n}{2}\right)\cos(n\pi t/T)\right)\cos(2\pi f_c t) \\ &\quad - \left(\sum_{n=1}^{\infty} \text{sinc}\left(\frac{n}{2}\right)\sin(n\pi t/T)\right)\sin(2\pi f_c t). \end{aligned}$$

3. (a)

$$\begin{aligned} R_y(t) &= \mathcal{F}\left(|H(f)|^2 N_0/2\right) \\ &= \text{sinc}^2(Bt) B N_0/2. \end{aligned}$$

(b) $y(t)$ is Gaussian with mean zero and variance $R_y(0) = BN_0/2$. So

$$f_Y(y) = \frac{1}{\sqrt{\pi B N_0}} e^{-\frac{y^2}{BN_0}}.$$

(c) Since $y(t)$ is Gaussian they are independent when $R_y(t) = 0$. This occurs when $t = \pm 1, \pm 2, \pm 3, \ldots$

4. (a)

$$\begin{aligned}
R_{X_c}(\tau) &= E(X_c(t+\tau)X_c(t)) \\
&= E\Big[\Big(X(t+\tau)\cos(2\pi f_0(t+\tau)) + \hat{X}(t+\tau)\sin(2\pi f_0(t+\tau))\Big) \\
&\quad \times \Big(X(t)\cos(2\pi f_0(t)) + \hat{X}(t)\sin(2\pi f_0(t))\Big)\Big] \\
&= E(X_c(t+\tau)X(t))\cos(2\pi f_0(t+\tau))\cos(2\pi f_0(t)) \\
&\quad +E(\hat{X}_c(t+\tau)\hat{X}(t+\tau))\sin(2\pi f_0(t+\tau))\sin(2\pi f_0(t)) \\
&\quad +E(X_c(t+\tau)\hat{X}(t))\cos(2\pi f_0(t+\tau))\sin(2\pi f_0(t)) \\
&\quad +E(\hat{X}_c(t+\tau)X(t))\sin(2\pi f_0(t+\tau))\cos(2\pi f_0(t)) \\
&= R_X(\tau)\cos(2\pi f_0(t+\tau))\cos(2\pi f_0(t)) \\
&\quad +R_{\hat{X}}(\tau)\sin(2\pi f_0(t+\tau))\sin(2\pi f_0(t)) \\
&\quad +R_{X\hat{X}}(\tau)\cos(2\pi f_0(t+\tau))\sin(2\pi f_0(t)) \\
&\quad +R_{\hat{X}X}(\tau)\sin(2\pi f_0(t+\tau))\cos(2\pi f_0(t)) \\
&= R_X(\tau)\cos(2\pi f_0(t+\tau))\cos(2\pi f_0(t)) \\
&\quad +R_X(\tau)\sin(2\pi f_0(t+\tau))\sin(2\pi f_0(t)) \\
&\quad -\hat{R}_X(\tau)\cos(2\pi f_0(t+\tau))\sin(2\pi f_0(t)) \\
&\quad +\hat{R}_X(\tau)\sin(2\pi f_0(t+\tau))\cos(2\pi f_0(t)) \\
&= R_X(\tau)\cos(2\pi f_0(\tau)) + \hat{R}_X(\tau)\sin(2\pi f_0(\tau))
\end{aligned}$$

where we have used the relations

$$\begin{aligned}
R_X(\tau) &= R_{\hat{X}}(\tau) \\
R_{X\hat{X}}(\tau) &= -\hat{R}_X(\tau) \\
R_{\hat{X}X}(\tau) &= R_{X\hat{X}}(-\tau) \\
&= -R_{X\hat{X}}(\tau) \\
&= \hat{R}_X(\tau).
\end{aligned}$$

Taking the Fourier transform and noting that $S_X(f) = S_{\hat{X}}(f)$ we have

$$\begin{aligned}
S_{X_c}(f) &= S_X(f-f_0)/2 + S_X(f+f_0)/2 - \mathrm{sgn}(f-f_0)S_X(f-f_0)/2 + \mathrm{sgn}(f+f_0)S_X(f+f_0)/2 \\
&= \begin{cases} S_X(f-f_0) & f < -f_0 \\ S_X(f-f_0) + S_X(f+f_0) & -f_0 < f < f_0 \\ S_X(f+f_0) & f > f_0. \end{cases} \\
&= \begin{cases} S_X(f-f_0) + S_X(f+f_0) & -f_0 < f < f_0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

where the last step follows because $X(t)$ is a bandpass process.

(b) The derivation is completely analogous to (a).

5. We first calculate what happens to the message signal as it passes from A to I.

At A

$$m(t)\cos 2\pi f_c t - \hat{m}(t)\sin 2\pi f_c t.$$

At B

$$\begin{aligned} A_0 m(t)\cos(2\pi f_c t)\cos(2\pi f_c t + \theta_c) &- A_0\hat{m}(t)\sin(2\pi f_c t)\cos(2\pi f_c t + \theta_c) \\ &= A_0 m(t)(\cos(4\pi f_c t + \theta_c) + \cos\theta_c)/2 \\ &- A_0\hat{m}(t)(\sin(4\pi f_c t + \theta_c) - \sin\theta_c)/2 \end{aligned}$$

At C

$$1/2 A_0 m(t)\cos\theta_c + 1/2 A_0\hat{m}(t)\sin\theta_c$$

At F

$$\begin{aligned} A_0 m(t)\cos(2\pi f_c t)\sin(2\pi f_c t + \theta_c) &- A_0\hat{m}(t)\sin(2\pi f_c t)\sin(2\pi f_c t + \theta_c) \\ &= A_0 m(t)(\sin(4\pi f_c t + \theta_c) + \sin\theta_c)/2 \\ &- A_0\hat{m}(t)(-\cos(4\pi f_c t + \theta_c) + \cos\theta_c)/2 \end{aligned}$$

At G

$$1/2 A_0 m(t)\sin\theta_c - 1/2 A_0\hat{m}(t)\cos\theta_c$$

At H

$$1/2 A_0\hat{m}(t)\sin\theta_c + 1/2 A_0 m(t)\cos\theta_c$$

At I

$$A_0(m(t)\cos\theta_c + \hat{m}(t)\sin\theta_c)$$

If we then note that $m(t)$ and $\hat{m}(t)$ are orthogonal and have the same power, we see that the power of the signal at I is

$$A_0^2 P_m.$$

To track the noise through the system write

$$n(t) = n_c(t)\cos(2\pi f_c t + \theta_c) - n_s(t)\sin(2\pi f_c t + \theta_c).$$

Then applying the analysis from above we have at C

$$1/2 A_0 n_c(t).$$

At G

$$-1/2 A_0 n_s(t).$$

At H

$$-1/2 A_0 \hat{n}_s(t).$$

At I

$$A_0/2(n_c(t) - \hat{n}_s(t).$$

Because $n_c(t)$ and $n_s(t)$ are independent, this has power

$$\begin{aligned} A_0^2/4(2P_n) &= A_0^2/4(4BN_0/2) \\ &= BN_0A_0^2/2. \end{aligned}$$

So the SNR is

$$\frac{2P_m}{BN_0}$$