EE126: Probability and Random Processes	**F'10**
Final — December 17	
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Formulas: In the unlikely event you forgot them since the last midterm, here are a few potentially useful formulas:

$$\mathbf{X} = N(\mu, \Sigma) \Leftrightarrow f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\{-\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\}$$

The LLSE of $\mathbf{X}$ given $\mathbf{Y}$ is $L[\mathbf{X}|\mathbf{Y}] = E(\mathbf{X}) + \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{Y}}^{-1}(\mathbf{Y} - E(\mathbf{Y}))$

$\text{cov}(\mathbf{AX}, \mathbf{BY}) = \mathbf{A}\text{cov}(\mathbf{X}, \mathbf{Y})\mathbf{B}^T$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $V = N(0, 1)$, then $P(V > 1) = 0.159, P(V > 1.64) = 0.05,$
$P(V > 1.96) = 0.025, P(V > 2) = 0.023, P(V > 2.58) = 0.005.$

Problem 1. *(***Short Problems 20%***)*

- *Complete the sentence: A random variable is*

- *Let X, Y be i.i.d. N(0, 1). Using the characteristics function, show that $Z = X + Y$ is Gaussian.*

- *Let X, Y be i.i.d. Exp(1). Derive the p.d.f. of $Z = X + Y$.*

- *Let X be a random variable that takes values in $\{0, 1, 2, 3, \ldots\}$ and is such that*

$$P[X \geq n + m \mid X \geq n] = P(X \geq m), \forall m, n \geq 0.$$

What are the possible p.m.f.'s of X?

- *Let X, Y be i.i.d. U[-1, 1]. What is the LLSE $L[X^2|X + Y]$?*

- *Give an example of a null recurrent irreducible discrete time Markov chain.*

Problem 2. *(15%) Assume that, given* $X = x$*,* $Y = N(2x + 3, 5)$*. Assume also that* $X = Exp(1)$*.*

(a) What is $MLE[X \mid Y]$*?*

(b) What is $MAP[X \mid Y]$*?*

Problem 3. *(**20%**)*

The random vector (X, Y) is picked uniformly in the quarter circle

$$C := \{(x, y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}.$$

a) Sketch C and draw a guess for the LLSE $L[X|Y]$. Explain your guess in a few words. (The second part asks for a calculation, but a guess should help you check that result.)

b) Calculate $L[X \mid Y]$.

Problem 4. *(**15%**) A monkey types random keys on a keyboard. Assume that each key that he types is equally likely to be any one of the 26 letters or a space. (We assume that there are no other symbols on the keyboard.) The monkey types 100 keys per second. In this problem, we explore the average time until the monkey types the symbols of "I WANT A BANANA" consecutively.*

a) Consider the following approximation. With some probability α, the monkey succeeds in 15 key strokes. With probability $1-\alpha$, the monkey has wasted some time T and has to try again, from scratch. What is then the average time A required with this approximation? What is α and what is a sensible value for T?

b) Formulate the problem as the first hitting time of a Markov chain. [Hint: Be careful of what happens when the monkey types "I".]

c) Write the first step equations for the mean value of the hitting time of the Markov chain, with the appropriate boundary conditions. (Do not solve.)

Problem 5. *(15%)*

You are given three light bulbs with i.i.d. exponentially distributed life times with mean 1 year.

a) You turn one light bulb on until it burns out, then turn on the second one. What is the probability that the two bulbs burn out before one year?

b) You turn the three light bulbs on at the same time. What is the probability that at least two bulbs burn out before one year? [Hint: Let V be the time when the first bulb burns out and Y be the time when the second one burns out. What is the p.d.f. of V? Using the memoryless property of the exponential distribution, you can obtain easily the distribution of $Y - V$.]

Problem 6. *(15%) A target has location* $\mathbf{X} = N(0, \Sigma)$ *in* $\Re^2$. *Your radar indicates that the target is at location* $\mathbf{Y} = \mathbf{X} + \mathbf{Z}$ *where* $\mathbf{Z} = N(0, \sigma^2 \mathbf{I})$ *and is independent of* $\mathbf{X}$. *You want to detonate a bomb that destroys the target if it explodes within a radius* r *of the target.*

a) Where should you place the bomb to maximize the probability of destroying the target?

b) Particularize the result to the case where

$$\Sigma = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \text{ and } \sigma^2 = \frac{1}{4}.$$