QUESTION BOOKLET
EE 126 Spring 2006 Final Exam
Wednesday, May 17, 8am–11am

DO NOT OPEN THIS QUESTION BOOKLET UNTIL YOU ARE TOLD TO DO SO

- You have 180 minutes to complete the final.
- The final consists of five (FIVE) problems, provided in the question booklet (THIS BOOKLET), that are in no particular order of difficulty.
- Write your solution to each problem in the space provided in the solution booklet (THE OTHER BOOKLET). Try to be neat! If we can't read it, we can't grade it.
- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely explain your reasoning and show all relevant work. The grade on each problem is based on our judgment of your understanding as reflected by what you have written.
- This is a closed-book exam except for three sheets of handwritten notes, plus a calculator.

Problem 1: (22 points)

Suppose that a pair of random variables X and Y has a joint PDF that is uniform over the shaded region shown in the figure below:

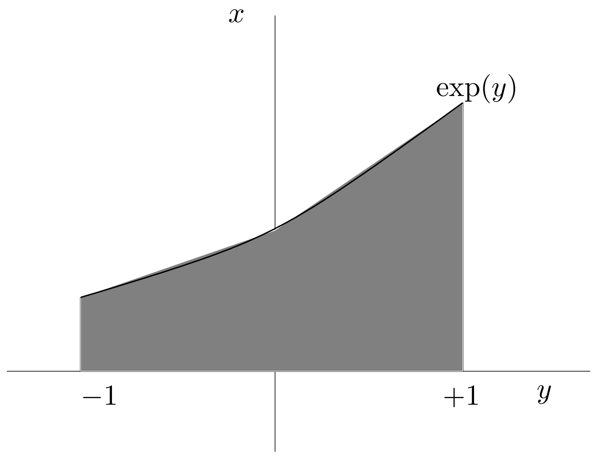


Figure 1: Joint PDF of random variables X and Y.

(a) (7 pt) Compute the Bayes' least squares estimator (BLSE) of X based on Y. (**Note:** You should evaluate the required integrals; however, your answer can be left in terms of quantities like $1/e$ or $\sqrt{2}$).

(b) (7 pt) Compute the linear least squares estimator (LLSE) of X based on Y, as well as the associated error variance of this estimator. Is the LLSE the same as the BLSE in this case? Why or why not? (**Note:** You should evaluate the required integrals; however, your answer can be left in terms of quantities like $1/e$ or $\sqrt{2}$).

(c) (8 pt) Now suppose that in addition to observing some value $Y = y$, we also know that $X \leq 1/e$. Compute the BLSE and LLSE estimators of X based on both pieces of information. Are the estimators the same or different? Explain why in either case.

Solution:

(a) The joint PDF has the form

$$f_{X,Y}(x,y) \quad = \quad \begin{cases} (1/C) & \text{if } |y| \leq 1 \text{ and } 0 \leq x \leq \exp(y) \\ 0 & \text{otherwise,} \end{cases}$$

where the normalization constant of the joint PDF is given by

$$\begin{aligned} C &= \int_{-1}^{-1} \int_{0}^{\exp y} dx dy \\ &= \exp(y) \mid_{-1}^{+1} = e - 1/e. \end{aligned}$$

We have

$$f_Y(y) \;=\; (1/C) \int_0^{\exp(y)} \;=\; (1/C)\; \exp(y) \quad \text{for } |y| \leq 1.$$

Hence, for $|y| \leq 1$, the conditional distribution takes the form

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \exp(-y) \quad \text{for } \exp \leq x \leq \exp(y) \end{aligned}$$

Consequently, we have

$$\widehat{X}_{BLSE} \;=\; \mathbb{E}[X \mid Y] \;=\; \frac{1}{2}\exp(y).$$

(b) To compute the LLSE, we first compute

$$\begin{aligned} \mathbb{E}[Y] &= \frac{1}{C}\int_{-1}^{+1} y\exp(y)dy \\ &= \frac{1}{C}\frac{2}{e}, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[Y^2] &= \frac{1}{C}\int_{-1}^{+1} y^2\exp(y)dy \\ &= \frac{1}{C}\left[y^2\exp(y)\mid_{-1}^{+1} -2\int_{-1}^{-1} y\exp(y)dy\right] \\ &= \frac{1}{C}\left[e-(1/e)\right]-2\mathbb{E}[Y] \end{aligned}$$

Putting these pieces together yields the variance $\text{var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$.

Using iterated expectation, we compute

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[\mathbb{E}[X \mid Y]] \\ &= \frac{1}{2C}\int_{-1}^{+1} \exp(2y)dy \\ &= \frac{1}{4C}(e^2 - 1/e^2). \end{aligned}$$

Lastly, we compute

$$\begin{aligned} \mathbb{E}[XY] &= \frac{1}{C}\int_{-1}^{+1}\left(\int_0^{\exp(y)} xydx\right)dy \\ &= \frac{1}{C}\int_{-1}^{+1}\left(\frac{x^2}{2}\mid_0^{\exp(y)}\right)ydy \\ &= \frac{1}{2C}\int_{-1}^{+1} y\exp(2y)dy \\ &= \frac{1}{8C}\left(e^2+\frac{3}{e^2}\right) \end{aligned}$$

which yields the covariance $\text{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$. With these pieces, the LLSE is given by

$$\widehat{X}_{LSSE} \;=\; \mathbb{E}[X] + \frac{\text{cov}(X,Y)}{\text{var}(Y)}\left[Y - \mathbb{E}[Y]\right].$$

In this case, the LLSE and BLSE are different: the BLSE is the optimum predictor of *any* function Y, and it is non-linear in this case. In contrast, the LLSE is restricted to linear functions of the data Y.

(c) Let $Z = (X \mid \{X \leq 1/e\})$. We compute

$$\begin{aligned}
\mathbb{P}[Z \leq z, Y \leq y] &= \frac{\mathbb{P}[Z \leq z, X \leq 1/e, Y \leq y]}{\mathbb{P}[X \leq 1/e]} \\
&= \mathbb{I}[z \leq 1/e]\,\frac{\mathbb{P}[X \leq z, Y \leq y]}{\mathbb{P}[X \leq 1/e]}
\end{aligned}$$

so that (Y, Z) is simply uniform over the box $\{|y| \leq 1,\ 0 \leq z \leq 1/e\}$.

Hence, we have

$$\begin{aligned}
\mathbb{E}[X \mid Y, (X \leq 1/e)] &= \mathbb{E}[Z \mid Y] \\
&= \frac{1}{2e}
\end{aligned}$$

for all $y \in [-1, +1]$. This BLSE is already linear, so it must be equivalent to the LLSE.

Problem 2 (18 points):

Bob the Gambler: Bob is addicted to gambling, and does so frequently. The time between any two consecutive visits that Bob makes to the local casino can be modeled as an exponentially-distributed random variable X with mean of 1 day. The times between different pairs of consecutive visits are independent random variables. Every time $i = 1, 2, 3\ldots$ that Bob gambles, he wins/loses a random amount of money modeled as a Gaussian random variable Y_i with mean 0 and variance σ^2. The amounts of money that he wins/loses on different occasions are independent random variables. Suppose that Bob starts off at time $t = 0$ with 0 dollars.

(a) (3 pt) What is the distribution of $N(t)$, the number of times Bob goes to the casino before some fixed time $t \geq 0$?

(b) (2 pt) What is PDF of $M(t)$, the amount of money Bob has won/lost by time t?

(c) (5 pt) Using Chernoff bound, bound the probability that $M(t)$ is greater that a dollars. (**Note:** You do NOT need to solve the minimization problem in the Chernoff bound.)

(d) (6 pt) For a given $t > 0$, define a random variable

$$Z_t = \frac{M(t)}{t},$$

corresponding to the average amount of money that Bob has won/lost at time t. As $t \to \infty$, does Z_t converge in probability? If so, prove it. If not, explain intuitively why not.

Solution:

(a) (2pt) $N(t)$ is a Poisson distribution with parameter t

(b) (2pt) Since $M(t) = \sum_{i=1}^{N(t)} Y_i$, conditioning over $N(t)$ we have

$$f_{M(t)}(m) = \sum_{j=0}^{\infty} \frac{t^j}{j!} e^{-t} \frac{1}{\sqrt{2\pi j\sigma^2}} e^{-\frac{m^2}{2j\sigma^2}}$$

(c) (5pt) Since $M_{M(t)}(s) = M_{N(t)}(s)\big|_{e^s = M_Y(s)} = e^{t \cdot e^{\sigma^2 s^2/2} - 1}$, we have

$$P(M(t) > a) \leq e^{-\max_{s \geq 0} sa - e^{\sigma^2 s^2/2}}$$

(d) (5pt) One would expect that $Z_t \to 0$ in probability. To show this, we use Chebyshev inequality to bound

$$P\left(\left|Z_t - \frac{1}{t}\mathbb{E}[M(t)]\right| > \epsilon\right) = P\left(\left|\frac{1}{t}\sum_{i=1}^{N(t)} Y_i\right| > \epsilon\right)$$

Where we have used $\mathbb{E}[M(t)] = \mathbb{E}_N[\mathbb{E}[M(t)|N = n]]$ and $\mathbb{E}[M(t)|N = n] = 0$ by symmetry. First we compute

$$\begin{aligned}
\text{var}\left(\sum_{i=1}^{N(t)} Y_i\right) &= E\left[\text{var}\left(\sum_{i=1}^{N(t)} Y_i|N(t)\right)\right] + \text{var}\left[E\left(\sum_{i=1}^{N(t)} Y_i|N(t)\right)\right] \\
&= \mathbb{E}[N(t)\sigma^2] \\
&= t\sigma^2
\end{aligned}$$

so

$$\begin{aligned}
P\left(\left|\frac{1}{t}\sum_{i=1}^{N(t)} Y_i\right| > \epsilon\right) &\leq \frac{\text{var}\left(\frac{1}{t}\sum_{i=1}^{N(t)} Y_i\right)}{\epsilon^2} \\
&\leq \frac{t\sigma^2}{t^2\epsilon^2} \\
&\to 0
\end{aligned}$$

when $n \to \infty$

Problem 3: (20 points)

True or false: For each of the following statements, either give a counterexample to show that it is false, or provide an argument to justify that it is true. (**Note:** You will receive no points for just guessing the correct answer; full points will be awarded only when an answer is justified with an example or argument.)

(a) (4 pt) If the Bayes' least squares estimator of X given Y is equal to $\mathbb{E}[X]$, then X and Y are independent.

(b) (4 pt) If random variables X and Y are independent, then they are conditionally independent given any other random variable Z.

(c) (4 pt) Given a sequence of random variables $\{X_n\}$ such that $\mathbb{E}[X_n] = n$ and $\mathbb{E}[X_n^2] = (n+1)^2$, the sequence $Y_n := \frac{X_n}{n}$ converges in probability to some real number.

(d) (4 pt) If the linear-least squares estimator $\widehat{X}_{LLSE}$ and Bayes' least-squares estimator $\widehat{X}_{BLSE}$ (of X based on Y) are equal, then the random variables X and Y must be jointly Gaussian.

(e) (4 pt) There do not exist any pairs of random variables X and Y with $\mathbb{E}[X^4] = 4$, $\mathbb{E}[Y^4] = 1$ and $\mathbb{E}[X^2 Y^2] = 3$.

Solutions:

(a) FALSE. Let $X \sim N(0,1)$, and let $Y = ZX$, where Z is equal to -1 and $+1$ with probability $\frac{1}{2}$. Then conditioned on $Y = y$, X is equal to $-y$ and $+y$ with probability $\frac{1}{2}$, so that $\mathbb{E}[X \,|Y = y] = 0 = \mathbb{E}[X]$. However, X and Y are certainly not independent.

(b) FALSE. Let X and Y be independent Bernoulli RVs, and let $Z = X + Y$. Given $Z = 0$, we have $X = Y$.

(c) TRUE. In fact, Y_n converges in probability to 1. We compute $\mathbb{E}[Y_n] = 1$ and $\text{var}(Y_n) = \frac{2n+1}{n^2}$. Applying Chebyshev's inequality yields

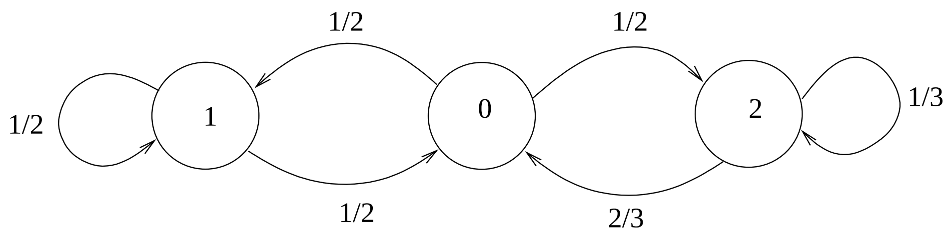
$$\mathbb{P}[|Y_n - 1| \geq \epsilon] \quad \leq \quad \frac{\text{var}(Y_n)}{\epsilon^2} \to 0$$

as $n \to +\infty$.

(d) FALSE. Consider part (c) of problem 1 (this exam).

(e) TRUE. We know that $\mathbb{E}[(X^2 - Y^2)^2]$ must be non-negative. We can expand it to $\mathbb{E}[X^4] - 2\mathbb{E}[X^2 Y^2] + \mathbb{E}[Y^4]$. If we plug in the numbers provided, we get $4 - 2 * 3 + 1 = -1$, but since our original expression had to be non-negative, we see that the provided values are not possible.

Problem 4: (20 points) The position of a moving particle can be modeled by a Markov chain on the states $\{0, 1, 2\}$ with the state transition diagram shown below



(a) (3 pt) Classify the states in this Markov chain. Is the chain periodic or aperiodic?

(b) (3 pt) In the long run, what fraction of time does the particle spend in state 1?

(c) (4 pt) Suppose that the starting position X_0 is chosen according to the the steady state distribution. Conditioned on $X_2 = 2$, what is the probability that $X_0 = 0$?

(d) (5 pt) Suppose that $X_0 = 0$, and let T denote the first time by which the particle has visited all the states. Compute $\mathbb{E}[T]$. [*Hint*: This problem is made easier by a suitable choice of conditioning.]

(e) (5 pt) Suppose now we have two particles, both independently moving according to the Markov chain. One particle starts in state 2, and the other particle starts in state 1. What is the average time before at least one of the particles is in state 0?

Solution:

(a) (2pt) The Markov chain has one recurrent aperiodic class.

(b) (2pt) Solving

$$\begin{aligned} \pi_1 &= \pi_0 \\ \frac{1}{2}\pi_0 &= \frac{2}{3}\pi_2 \\ \pi_0 + \pi_1 + \pi_2 &= 1 \end{aligned}$$

we get $\pi_0 = \frac{4}{11}$, $\pi_1 = \frac{4}{11}$, $\pi_2 = \frac{3}{11}$.

(c) (2pt) From Bayes rule

$$P(X_0 = 0|X_2 = 2) = \frac{P(X_2 = 2|X_0 = 0)P(X_0 = 0)}{P(X_2 = 2)}$$

where by assumption $P(X_0 = 0) = \pi_0$ and, from total probability theorem

$$\begin{aligned} P(X_2 = 2|X_0 = 0) &= \sum_i P(X_2 = 2|X_0 = 0, X_1 = i)P(X_1 = i|X_0 = 0) \\ &= \sum_i P(X_2 = 2|X_1 = i)P(X_1 = i|X_0 = 0) \\ &= \sum_i p_{i2}p_{0i} \\ &= \frac{1}{6}. \end{aligned}$$

Similarly

$$\begin{aligned} P(X_2 = 2) &= \sum_i P(X_2 = 2|X_0 = i)\pi_i \\ &= \sum_i \sum_j P(X_2 = 2|X_1 = j, X_0 = i)P(X_1 = j|X_0 = i)\pi_i \\ &= \sum_i \sum_j p_{j2}p_{ij}\pi_i \\ &= \frac{1}{6}\pi_0 + \frac{1}{4}\pi_1 + \left(\frac{2}{3}\frac{1}{2} + \frac{1}{3}\frac{1}{3}\right)\pi_2. \\ &= \frac{3}{11} \end{aligned}$$

So, $P(X_0 = 0|X_2 = 2) = \frac{2}{9}$

(d) (5pt) Conditioning on the first move of the particle,

$$\mathbb{E}[T] = \frac{1}{2}[\mathbb{E}[T|X_1 = 1] + \frac{1}{2}[\mathbb{E}[T|X_1 = 2]$$

where $\mathbb{E}[T|X_1 = 1]$ is the mean first passage time to state 2 starting from state 1 plus one (to account for the first step to 1), and $\mathbb{E}[T|X_1 = 2]$ is the mean first passage time to state 1 starting from state 2 plus one (to account for the first step to 2). From the first step analysis, one can write two systems of equations, the first of which is given below

$$t_1 = 1 + \frac{1}{2}t_0 + \frac{1}{2}t_1$$
$$t_0 = 1 + \frac{1}{2}t_1 + \frac{1}{2}t_2$$

Here we are using the book's notation, with t_i denoting the mean first passage time from state i to state 2 (the 2 is not made explicit). Therefore, $t_2 = 0$. We can solve the system to get $t_1 = 6$, or $\mathbb{E}[T|X_1 = 1] = t_1 + 1 = 7$. Now for the second system:

$$t_2 = 1 + \frac{2}{3}t_0 + \frac{1}{3}t_2$$
$$t_0 = 1 + \frac{1}{2}t_1 + \frac{1}{2}t_2$$

Here we have $t_1 = 0$, and we can solve to get $t_2 = 5$ or $\mathbb{E}[T|X_1 = 2] = 6$. Therefore our final answer is $\frac{1}{2} \cdot 6 + \frac{1}{2} \cdot 7 = \frac{13}{2}$.

(e) (3pt) The problem asks to compute $\mathbb{E}[Z]$ where $Z = \min(X_1, X_2)$ and X_1 and X_2 are geometric random variables with parameter 1/2 and 2/3 respectively. First, let us compute the CDF of a geometric random variable X with parameter p:

$$\begin{aligned}
P(X > z) &= \sum_{k=z+1}^{\infty} p(1-p)^{k-1} \\
&= 1 - \sum_{k=1}^{z} p(1-p)^{k-1} \\
&= 1 - p\sum_{k=0}^{z-1}(1-p)^k \\
&= 1 - p\frac{1-(1-p)^z}{1-(1-p)} \\
&= (1-p)^z
\end{aligned}$$

Hence,

$$\begin{aligned}
P(Z > z) &= P(\min(X_1, X_2) > z) \\
&= P(X_1 > z)P(X_2 > z) \\
&= \left(1 - \frac{1}{2}\right)^z \left(1 - \frac{2}{3}\right)^z \\
&= \left(\frac{1}{6}\right)^z
\end{aligned}$$

And

$$\begin{aligned}
\mathbb{E}[Z] &= \sum_{z=0}^{\infty} P(Z > z) \\
&= \frac{1}{1 - \frac{1}{6}} \\
&= \frac{6}{5}
\end{aligned}$$

Problem 5: (20 points)

Action figure collector: Acme Brand Chocolate Eggs are hollow on the inside, and each egg contains an action figure, chosen uniformly from a set of four different action figures. The price of any given egg (in dollars) is an exponentially distributed random variable with parameter λ, and the prices of different eggs are independent. For $i = 1, \ldots, 4$, let T_i be a random variable corresponding to the number of eggs that you purchase in order to have i different action figures. To be precise, after purchasing T_2 eggs (and not before), you have at least one copy of exactly 2 different action figures; and after purchasing T_4 eggs (and not before), you have at least one copy of all four action figures.

(a) (2 pt) What is the PMF, expected value, and variance of T_1?

(b) (4 pt) What is the PMF and expected value of T_2?

(c) (5 pt) Compute $\mathbb{E}[T_4]$ and $\text{var}(T_4)$. (*Hint:* Find a representation of T_4 as a sum of independent RVs.)

(d) (5 pt) Compute the moment generating function of T_4.

(e) (4 pt) You keep buying eggs until you have collected all four action figures. Letting Z be a random variable representing the total amount of money (in dollars) that you spend, compute $\mathbb{E}[Z]$ and $\text{var}[Z]$.

Solution:

(a) $P(T_1 = 1) = 1$ and $P(T_1 = i) = 0$ $i \neq 1$. Thus $\mathbb{E}[T_1] = 1$ and $\text{var}(T_1) = 0$

(b) $T_2 = T_1 + X_2$ where X_2 is a geometric(3/4) r.v. So $P(T_2 = k) = \frac{3}{4}\left(\frac{1}{4}\right)^{k-1}$ $k = 2, 3, \ldots$ and $\mathbb{E}[T_2] = \mathbb{E}[T_1] + \mathbb{E}[X_2] = 1 + \frac{4}{3} = \frac{7}{3}$

(c) $T_4 = 1 + X_2 + X_3 + 4$ where X_3 is a geometric(1/2) r.v and X_4 is a geometric(1/4) r.v.. From this, one get

$$\mathbb{E}[T_4] = \mathbb{E}[1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \mathbb{E}[X_4] = \frac{25}{3}$$

and

$$\text{var}[T_4] = \text{var}[1] + \text{var}[X_2] + \text{var}[X_3] + \text{var}[4] = \frac{130}{9}$$

(d) From

$$M_{T_4}(s) = e^s M_{X_2}(s) M_{X_3}(s) M_{X_4}(s)$$

we get

$$M_{T_4}(s) = e^s \left(\frac{\frac{3}{4}e^s}{1 - \frac{1}{4}e^s}\right)\left(\frac{\frac{1}{2}e^s}{1 - \frac{1}{2}e^s}\right)\left(\frac{\frac{1}{4}e^s}{1 - \frac{3}{4}e^s}\right)$$

(e) Since $Z = \sum_{i=1}^{T_4} P_i$, where P_i is the price of egg i, we have:

$$\mathbb{E}[Z] = \mathbb{E}[\mathbb{E}[Z|T_4]] = \frac{1}{\lambda}\mathbb{E}[T_4] = \frac{25}{3\lambda}$$

and

$$\text{var}[Z] = \text{var}[\mathbb{E}[Z|T_4]] + \mathbb{E}[\text{var}[Z|T_4]] = \frac{1}{\lambda^2}\frac{130}{9} + \frac{25}{3}\frac{1}{\lambda^2} = \frac{205}{9\lambda^2}$$

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