Department of EECS - University of California at Berkeley

EECS 126 - Probability and Random Processes - Spring 2008

Midterm 1: 2/28/2007

Name (Last, First):

SID:

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1. (15%)

There are two coins. Coin 1 is fair. Coin 2 is such that $P(H) = 0.6$.

1) You flip the two coins together repeatedly. What is the probability that coin 1 yields H before coin 2?

2) You are given one of the two coins, with equal probabilities. You flip the coin twice and you get H both times. What is the probability that you got coin 1?

For part (1), first we define A to be the event that coin 1 yields a H before coin 2; now we are going to use conditional probability and the total probability law - we condition on the first flip. Let $C_i(j)$ denote the j^{th} flip of coin i. So we get

$$\begin{aligned} P(A|C_1(1) = H, C_2(1) = H) &= 0 \\ P(A|C_1(1) = H, C_2(1) = T) &= 1 \\ P(A|C_1(1) = T, C_2(1) = H) &= 0 \\ P(A|C_1(1) = T, C_2(1) = T) &= P(A) \end{aligned}$$

where the last equation comes because from flip to flip the coins are memoryless, and so flips across time are independent. Now we apply the total probability law.

$$\begin{aligned} P(A) &= \sum_{c_1,c_2} P(A|C_1(1) = c_1, C_2(1) = c_2)P(C_1(1) = c_1, C_2(1) = c_2) \\ &= 1 \cdot \frac{1}{2} \cdot \frac{2}{5} + P(A) \cdot \frac{1}{2} \cdot \frac{2}{5} \\ &= \frac{2}{10} + P(A) \cdot \frac{2}{10} \\ P(A) \cdot \frac{8}{10} &= \frac{2}{10} \\ P(A) &= \frac{1}{4} \end{aligned}$$

Another method of calculating this probability is to notice that

$$\begin{aligned}
P(A) &= \sum_{i=1}^{\infty} P(\text{first H we see is from coin 1 on the } i^{th} \text{ flip}) \\
&= \sum_{i=1}^{\infty} P(C_1(1) = T, C_1(2) = T, \ldots, C_1(i) = H, C_2(1) = T, \ldots C_2(i) = T) \\
&= \sum_{i=1}^{\infty} (0.5)^i (0.4)^i \\
&= \sum_{i=1}^{\infty} (0.2)^i \\
&= \frac{1}{1 - 0.2} - 1 \\
&= \frac{1}{4}
\end{aligned}$$

For part (2), we are looking for $P(C_1|HH)$, where C_1 is the event that we have coin 1, and HH is the event that the first two flips are heads. So, we apply Bayes' rule.

$$\begin{aligned}
P(C_1|HH) &= \frac{P(C_1, HH)}{P(HH)} \\
&= \frac{P(HH|C_1)P(C_1)}{P(HH|C_1)P(C_1) + P(HH|C_2)P(C_2)} \\
&= \frac{(1/2)^2 \cdot (1/2)}{(1/2)^2 \cdot (1/2) + (3/5)^2 \cdot (1/2)} \\
&= \frac{1/8}{1/8 + 9/50} \\
&= \frac{1/4}{1/4 + 9/25} \\
&= \frac{1}{4} \cdot \frac{100}{25 + 36} \\
&= \frac{25}{61}
\end{aligned}$$

2. (20%)

You throw a dart at a circular target with radius 1. You miss the target with probability 0.2. If you hit the target, the dart location is uniformly distributed inside the target. Let X be the distance from dart to the center of the target when you hit it and $X = 2$ when you miss the target.

1) What is the p.d.f. of X;

2) Plot the c.p.d.f. of X;

3) Calculate $Var(X)$, the variance of X.

For part (1), we recognize that X is actually the combination of a discrete random variable (if we miss) and a continuous random variable (if we hit the target); we'll first deal with the continuous part. So, lets first calculate the CDF given that we hit the target; supposing $0 \leq x \leq 1$,

$$\begin{aligned} P(X \leq x|\text{hit}) &= \frac{\pi x^2}{\pi \cdot 1^2} \\ &= x^2 \end{aligned}$$

We used in this part the fact that the location of the dart is located uniformly in the target, so it is just the ratio of the area of the circle of radius x to the area of the circle of radius 1.

So, the PDF is the derivative of this region; so given that we have a hit, for $0 \leq x \leq 1$, we get

$$f_X(x) = 2x$$

Also, we know that if $x = 2$, then the probability is going to be 0.2. We model that with a delta function. Also, we know that we hit with probability 0.8 and that the density calculated above is only valid for $0 \leq x \leq 1$, so we need to scale it and multiply it by an indictor variable to get

$$\begin{aligned} f_X(x) &= 0.8 \cdot 2x \cdot 1\{0 \leq x \leq 1\} + 0.2\delta(x-2) \\ &= 1.6x \cdot 1\{0 \leq x \leq 1\} + 0.2\delta(x-2) \end{aligned}$$

For part (2), I need to put a graph here.

For part (3), remember that the variance of X is

$$Var(X) = E[X^2] - E[X]^2$$

Lets calculate both of these values separately. Just use the PDF.

$$
\begin{aligned}
E[X^2] &= \int x^2 f_X(x)\,dx \\
&= \int_0^1 1.6x^3\,dx + 0.2 \cdot 2^2 \\
&= 0.4x^4|_0^1 + 0.8 \\
&= 1.2 \\
&= \frac{6}{5}
\end{aligned}
$$

$$
\begin{aligned}
E[X] &= \int x f_X(x)\,dx \\
&= \int_0^1 1.6x^2\,dx + 0.2 \cdot 2 \\
&= \frac{1.6}{3}x^3|_0^1 + 0.4 \\
&= \frac{1.6}{3} + frac1.23 \\
&= \frac{2.8}{3} \\
&= \frac{14}{15}
\end{aligned}
$$

$$
\begin{aligned}
Var(X) &= \frac{6}{5} - \frac{14^2}{15^2} \\
&= \frac{270 - 196}{225} \\
&= \frac{74}{225}
\end{aligned}
$$

3. (20%) You pick a point ω uniformly in the square $[0,1]^2$ and you designate the coordinates of the point by $X(\omega)$ and $Y(\omega)$.

1) Calculate $E(|X-Y|)$.

2) Calculate $P[X \le x \mid |X-Y| > 0.5]$ for $x \in [0,1]$.

For part (1), we notice that if $X \ge Y$, then $|X-Y| = X-Y$, and if $X \le Y$, then $|X-Y| = Y-X$; so we will split up our calculation of the expectation; also since the point is uniform in $[0,1]^2$, then the PDF is $f_{X,Y}(x,y) = 1$ as long as $0 \le x, y \le 1$. So, we get

$$\begin{aligned} E[|X-Y||X \ge Y] &= \int_0^1 \int_0^x (x-y) \cdot 1 \, dx \, dy \\ &= \int_0^1 x - \frac{x^2}{2} \, dx \, dy \\ &= \left(\frac{x^2}{2} - \frac{x^3}{6} \right)_0^1 \\ &= \frac{1}{2} - \frac{1}{6} \\ &= \frac{1}{3} \\ E[|X-Y||X \ge Y] &= \frac{1}{3} \end{aligned}$$

The last fact we deduced through symmetry. Therefore, we conclude that

$$E[|X-Y|] = \frac{1}{3}$$

For part (2), we first consider the regions where $|X-Y| > 0.5$; it is the two triangles denoted in the figures. The total area of these two figures is 0.25, so we know that $P(|X-Y| > 0.5)$ is 0.25.

So we get

$$P(X \le x||X-Y| > 0.5) = \frac{P(X \le x, |X-Y| > 0.5)}{P(|X-Y| > 0.5)}$$

In order to calculate the top, we recognize that our calculation will look different for $x < 0.5$ and $x > 0.5$; so for $x < 0.5$, the probability is the partial area of the left triangle.

$$\begin{aligned} P(X \le \hat{x}, |X-Y| > 0.5) &= \int_0^{\hat{x}} \int_{0.5+x}^1 dy \, dx \\ &= \int_0^{\hat{x}} 0.5 - x \, dx \\ &= 0.5\hat{x} - 0.5\hat{x}^2 \end{aligned}$$

For $x > 0.5$, the probability is the area of the left triangle added to the partial area of the right triangle. So we get

$$
\begin{aligned}
P(X \leq \hat{x}, |X-Y| > 0.5) &= 0.125 + \int_{0.5}^{\hat{x}} \int_{0}^{x-0.5} dy\, dx \\
&= 0.125 + \int_{0.5}^{\hat{x}} x - 0.5\, dx \\
&= 0.125 + 0.5\hat{x}^2 - 0.5\hat{x} - 0.125 + 0.25 \\
&= 0.5\hat{x}^2 - 0.5\hat{x} + 0.25
\end{aligned}
$$

So, plugging all of this back in, we get

$$
\begin{aligned}
P(X \leq x || X-Y| > 0.5) &= \frac{P(X \leq x, |X-Y| > 0.5)}{P(|X-Y| > 0.5)} \\
&= \begin{cases} 2x - 2x^2 & 0 \leq x \leq 0.5 \\ 2x^2 - 2x + 1 & 0.5 \leq x \leq 1 \\ 0 & o.w. \end{cases}
\end{aligned}
$$

4. (15%)

Assume that humanity will either survive 10 billion years or ten million years, with equal probabilities. For simplicity, assume that the population is constant and about equal to 8 billion people, in both cases. Assume also that you are picked randomly human, among all humans who will ever live. You observe that humanity has been around for about 5 million years. What is the probability that humanity will survive ten billion years, given your observation?

In this problem, we randomly pick a human among all humans who have ever lived. The total number of humans who ever live is either equal to $8B \times 10M$ or $8B \times 10B$, each with probability 1/2. We let P_{10M} (resp. P_{10B}) the event that the total humanity population is $10M$ (resp. $10B$). We also let E_{5M} denote the event that a randomly chosen individual belongs to the generation of $5M$. We have that $Pr(E_{5M}|P_{10M}) = \frac{8B}{8B\times 10M}$ and $Pr(E_{5M}|P_{10B}) = \frac{8B}{8B\times 10B}$.
In this problem, we would like to compute $Pr(P_{10B}|E_{5M})$. This can be done using Bayes' rule.

$$Pr(P_{10B}|E_{5M}) = \frac{Pr(E_{5M}|P_{10B})Pr(P_{10B})}{Pr(E_{5M}|P_{10B})Pr(P_{10B}) + Pr(E_{5M}|P_{10M})Pr(P_{10M})}$$

This gives

$$Pr(P_{10B}|E_{5M}) = \frac{10^{-9} \times .5}{10^{-9} \times .5 + 10^{-6} \times .5} = \frac{1}{1 + 10^3}$$

5. (15%)

A randomly picked 126 student has a 20% chance of being a genius and an 80% chance of being very smart but somehow short of genius. A genius gets a score on the first midterm that is uniformly distributed in $[70, 100]$. A very smart student gets a score that is uniformly distributed in $[0, 100]$. A genius has a probability 80% of going to graduate school and a very smart student has a probability 20% of going to graduate school. What is the probability that a randomly picked student who gets a score of 80 will go to graduate school?

Let S denote the event that a student is smart, G the event that she/he is genius, and Gr the event that she/he goes to grade school. Let also M_{80} be the event that a student gets an 80 in the midterm. We have that $P(S) = 4/5$, $P(G) = 1/5$, $P(M_{80}|S) = 1/101$, and $P(M_{80}|G) = 1/31$. Also $P(Gr|G) = 4/5$ and $P(Gr|S) = 1/5$.
Now we are asked to compute $P(Gr|M_{80})$.
Using the definition of conditional probability, we have

$$P(Gr|M_{80}) = \frac{P(Gr, M_{80})}{P(M_{80})}$$

Conditioning on S and G, we the equation above becomes

$$P(Gr|M_{80}) = \frac{P(Gr, M_{80}|S)P(S) + P(Gr, M_{80}|G)P(G)}{P(M_{80}|S)P(S) + P(M_{80}|S)P(S)}$$

Now given S (or G), Gr and M_{80} are independent and we can rewrite the above equation as

$$P(Gr|M_{80}) = \frac{P(Gr|S)P(M_{80}|S)P(S) + P(Gr|G)P(M_{80}|G)P(G)}{P(M_{80}|S)P(S) + P(M_{80}|S)P(S)}$$

This gives

$$P(Gr|M_{80}) = \frac{\frac{1}{5}\frac{1}{101}.5 + \frac{4}{5}\frac{1}{31}.5}{\frac{1}{101}.5 + \frac{1}{31}.5} = \frac{29}{44}$$

6. (15%)

Assume that $P(X = n) = (1-p)^{n-1}p, n \geq 1$ where $p \in (0,1)$. Calculate $E(X^k)$ for $k \geq 1$.

By definition $E(X^k)$ (the k'th moment) of a geometric random variable is given by

$$E(X^k) = \sum_{n=1}^{\infty} n^k p(1-p)^{n-1} = \frac{p}{1-p} \sum_{n=1}^{\infty} n^k (1-p)^n$$

In class we have computed the first moment $E(X) = \frac{1}{p}$. This is usually done by taking the derivative of the the series $\sum_{n=0}^{\infty} n^k a^n = \frac{1}{1-a}$ for $|a| < 1$. Let us generalize this to compute the k'th moment. For that we define

$$\alpha_k = \sum_{n=0}^{\infty} n^k a^n, \quad k = 0, 1, 2, \ldots$$

We know that $\alpha_0 = \frac{1}{1-a}$. Also notice that

$$\alpha_{k+1} = a \partial_a \alpha_k$$

By repeating this recursive equation (where at each step we compute the derivative with respect to a and multiply it by a), we can express α_k in terms of α_0 as follow

$$\alpha_k = \sum_{n=1}^{k} b_{n,k} a^n \partial_a^{(n)} \alpha_0$$

where the coefficients $b_{n,k}$ are computed recursively as

$$b_{0,0} = 1, \quad b_{1,k} = b_{k,k}, \ \forall k \tag{1}$$

$$b_{j+1,k} = b_{j,k} + (j+1) b_{j+1,k}, \ j = 2 \ldots k \tag{2}$$

We given below the values of $b_{j,k}$ for $k = 1, \ldots, 6$

```
1
1  1
1  3  1
1  7  6  1
1 15 25 10 1
1 31 80 65 15 1
..................
```

The derivatives of α_0 are given by

$$\partial_a^{(n)} \alpha_0 = \frac{1}{(1-a)^{k+1}}$$

Thus we have

$$\alpha_k = \sum_{n=1}^{k} \frac{b_{n,k} a^n}{(1-a)^{k+1}}$$

Replacing a by $1-p$ in the expression for α, we obtain

$$\sum_{n=1}^{\infty} n^k (1-p)^n = \sum_{n=1}^{k} \frac{b_{n,k}(1-p)^n}{p^{k+1}}$$

This gives

$$E(X^k) = \frac{1}{p^k} \sum_{n=1}^{k} b_{n,k}(1-p)^{n-1}$$

where the coefficients $b_{n,k}$ are given in 1 and 2.