EE126: Probability and Random Processes **SP'10**

Midterm — April 1

Lecturer: Jean C. Walrand SOLUTIONS *GSI: Baosen Zhang*

Formulas: Given the short attention span induced by twitter and the like, we thought you might appreciate not having to remember the following formulas. After all, they are on Wikipedia.

$$\mathbf{X} = N(\mu, \Sigma) \Leftrightarrow f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\{-\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\}$$

$$L[\mathbf{X}|\mathbf{Y}] = E(\mathbf{X}) + \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{Y}}^{-1}(\mathbf{Y} - E(\mathbf{Y}))$$

$$\text{cov}(\mathbf{AX}, \mathbf{BY}) = \mathbf{A}\text{cov}(\mathbf{X}, \mathbf{Y})\mathbf{B}^T.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$P(V > 1.64) = 0.05 \text{ when } V = N(0,1).$$

Problem 1. *(***Multiple Choice 20%***)*

- *If X, Y, Z are pairwise independent, then XY and Z are (circle the correct answer(s)): uncorrelated, independent, (possibly neither).*
- *Let X, Y, Z be general random variables. Then (circle the identities that always hold):*

$$E[L[X|Y]|Y] = E[X|Y]$$
$$(E[L[X|Y]|Y] = L[X|Y])$$
$$(L[L[X|Y,Z]|Y] = L[X|Y])$$
$$L[XY|Y] = YL[X|Y].$$

- *Jointly Gaussian random variables that are pairwise independent are also mutually independent (circle the correct answer):*
  (True), False.
- *The maximum of two independent exponentially distributed random variables is exponentially distributed (circle the correct answer):*
  True, (False).
- *Let X, Y be jointly Gaussian, zero mean, and unit variance random variables. Then (circle the statements that are certainly true):*

$$(\text{cov}(X,Y) \leq 1); \text{cov}(X,Y) \leq 0.5; P(X > Y) = 1/2; (L[X|Y] = E[X|Y].)$$

Problem 2. *(**Quick Calculations 20%**) Let X, Y, Z be i.i.d. $U[0,1]$. Calculate*

(a) $E[(X+Y)^2|X] = E[X^2 + 2XY + Y^2|X] = X^2 + 2XE(Y) + E(Y^2) = X^2 + X + 1/3$

(b) $E[(X+Y)(Y+Z)|Y] = E[XY + Y^2 + XZ + YZ|Y]$
$= Y/2 + Y^2 + 1/4 + Y/2 = Y^2 + Y + 1/4$

(c) $L[X+Y|X+Y+Z] = (2/3)(X+Y+Z)$, *by symmetry.*

(d) $L[(X+Y)Y|Y] = L[XY|Y] + L[Y^2|Y]$.
Now, $L[XY|Y] = E(X)Y = Y/2$. *Indeed,* $XY - Y/2 \perp Y$. *Also,*

$$L[Y^2|Y] = E(Y^2) + \frac{cov(Y^2, Y)}{\text{var}(Y)}(Y - E(Y)) = 1/3 + \frac{E(Y^3) - E(Y^2)E(Y)}{E(Y^2) - E(Y)^2}(Y - 1/2)$$
$$= 1/3 + \frac{1/4 - 1/6}{1/3 - 1/4}(Y - 1/2) = 1/3 + Y - 1/2 = Y - 1/6.$$

Hence, $L[(X+Y)Y|Y] = Y/2 + Y - 1/6 = 3Y/2 - 1/6$.

(e) $E[\cos(X+Y)|Y] = \int_0^1 \cos(x+Y)dx = \sin(1+Y) - \sin(Y)$.

Problem 3. *(**More Quick Calculations 20%**) Let X, Y, Z be i.i.d. $N(0,1)$. Calculate:*
[Hint: First calculate $\text{cov}(X+Y, X-Y)$.] *$cov(X+Y, X-Y) = 0 \Rightarrow X+Y \perp\!\!\!\perp X-Y$.*

(a) $E[\sin(X+Y)|X-Y] = E(\sin(X+Y))$, since $X+Y \perp\!\!\!\perp X-Y$
$= 0$, since the distribution of $X+Y$ is symmetric around 0.

(b) $E[2X+Y|X-Y] = E[X|X-Y]$ since $X+Y \perp\!\!\!\perp X-Y$
$= (X-Y)/2$, by symmetry.

(c) $E[(X+Y+Z)^2|X-Y] = E((X+Y+Z)^2)$ since $(X+Y, Z) \perp\!\!\!\perp X-Y$
$= var(X+Y+Z) = 3$.

(d) $E[X+2Y+3Z|X+Z, Y+2Z]$

$$= [4,8]\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}^{-1}\begin{bmatrix} X+Z \\ Y+2Z \end{bmatrix} = [4,8]\frac{1}{6}\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} X+Z \\ Y+2Z \end{bmatrix} = [2/3, 4/3]\begin{bmatrix} X+Z \\ Y+2Z \end{bmatrix}.$$

Problem 4. *(20%)*

(a) Write the projection characterization of $E[X|Y]$.

(b) Use this property to show that if Z *is independent of* (X, Y), *then* $E[Xg(Y)Z|Y] = E[X|Y]g(Y)E(Z)$.

(a) The projection characterization is that $E[X|Y]$ is the function of Y with the property that $X - E[X|Y] \perp h(Y), \forall h(.)$, i.e., $E((X - E[X|Y])h(Y)) = 0, \forall h(.)$.

(b) Since $V := E[X|Y]g(Y)E(Z)$ is a function of Y, to show that $V = E[Xg(Y)Z|Y]$, it suffices to check that $E((Xg(Y)Z - V)h(Y)) = 0, \forall h(.)$. Now,

$$E((Xg(Y)Zh(Y)) = E(Xg(Y)h(Y))E(Z), \text{ since } Z \perp\!\!\!\perp (X, Y)$$

and

$$E(Vh(Y)) = E(E[X|Y]g(Y)E(Z)h(Y)) = E(Z)E(g(Y)h(Y)E[X|Y])$$
$$= E(Z)E(E[Xg(Y)h(Y)|Y]) = E(Z)E(Xg(Y)h(Y))$$

where the last identity comes from the fact that $E(E[V|W]) = E(V)$ for any random variables V, W.

Problem 5. *(20%) When $X = 0, \mathbf{Y} = N(0, \Sigma)$ and when $X = 1, \mathbf{Y} = N(\mu, \Sigma)$ where*

$$\mu = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}^2 = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}.$$

(a) Find $MLE[X|\mathbf{Y}]$ and express it in the form $MLE[X|\mathbf{Y}] = 1\{\mathbf{a}^T\mathbf{Y} \geq \alpha$.

(b) Find $\hat{X}$ based on Y and taking values in $\{0, 1\}$ that maximizes $P[\hat{X} = 1|X = 1]$ subject to $P[\hat{X} = 1|X = 0] \leq 0.05$.

First we calculate

$$l(\mathbf{y}) = log(\frac{f_1(\mathbf{y})}{f_0(\mathbf{y})}).$$

We have

$$2.l(\mathbf{y}) = \mathbf{y}^T\Sigma^{-1}\mathbf{y} - (\mathbf{y} - \mu)^T\Sigma^{-1}(\mathbf{y} - \mu) = 2\mu^T\Sigma^{-1}\mathbf{y} - \mu^T\Sigma^{-1}\mu.$$

(a) $MLE[X|Y] = 1\{l(\mathbf{Y}) \geq 0\} = 1\{2\mu^T\Sigma^{-1}\mathbf{Y} \geq \mu^T\Sigma^{-1}\mu = 1\{\mathbf{a}^T\mathbf{Y} \geq \alpha\}$ where $\mathbf{a}^T = 2\mu^T\Sigma^{-1} = [34, 22]$ and $\alpha = \mu^T\Sigma^{-1}\mu = 61$.

(b) The solution is $\hat{X} = 1\{l(\mathbf{y}) > l_0\}$ where l_0 is such that $P[l(\mathbf{Y}) > l_0|X = 0] = 0.05$. Thus,

$$\hat{X} = 1\{\mathbf{a}^T\mathbf{Y} > c\}$$

where c is such that $P[\mathbf{a}^T\mathbf{Y} > c|X = 0] = 0.05$. Now, when $X = 0$, $Z := \mathbf{a}^T\mathbf{Y} = N(0, \sigma^2)$ where

$$\sigma^2 = \mathbf{a}^T\Sigma\mathbf{a} = 4\mu^T\Sigma^{-1}\Sigma\Sigma^{-1}\mu = 4\mu^T\Sigma^{-1}\mu = 244.$$

Thus, $P(Z > c) = P(V > c/\sqrt{244})$ where $V = N(0, 1)$. Thus, $P(Z > 1.64\sqrt{244}) = P(V > 1.64) = 0.05$. Hence,

$$\hat{X} = 1\{[34, 22]\mathbf{Y} > 1.64\sqrt{244}\}.$$