EECS 126 Probability and Random Processes University of California, Berkeley: Spring 2015
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Midterm Exam

Last name	First name	SID

Rules.

- You have 80 mins (5:10pm - 6:30pm) to complete this exam.
- The exam is not open book, but you are allowed one sheet of handwritten notes; calculators will be allowed.
- No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.

Please read the following remarks carefully.

- Show all work to get any partial credit.
- Take into account the points that may be earned for each problem when splitting your time between the problems.

Problem	Points earned	out of
Problem 1		25
Problem 2		20
Problem 3		25
Problem 4		30
Total		100

Problem 1[25]

(a) [6] The following questions pertain to a finite state MC that consists of a single periodic class (i.e, peiod > 1): explain your answer — a T/F answer won't get credit.

(i) [2] Do all the states have to be recurrent?

(ii) [2] Is it possible for any of the states to have self transitions?

(iii) [2] What does the solution to the balance equations tell you about the states of the MC?

(b) [7] A four state Disrete time MC with states $0, 1, 2, 3$ has the transition matrix

$$P = \begin{pmatrix} 0.2 & 0.3 & 0.5 & 0 \\ 0.2 & 0.1 & 0.3 & 0.4 \\ 0.4 & 0.6 & 0 & 0 \\ 0.1 & 0.9 & 0 & 0 \end{pmatrix}$$

Argue that this this MC consists of a single recurrent class. Assume that it is in steady state after running for $n > 500$ steps. Find the followingin terms of π_i's. (You don't have calculate the steady state distribution!)

(i) [3] $P(X_{1002} = 2 | X_{1000} = 0)$

(ii) [4] $P(X_{1000} = 0 | X_{1002} = 2)$

(c) [6] The freeway system consists of a very long entry ramp and a road. A pacing light regulates the entry of cars from ramp to road. There are multiple exits.

(i) [3] Cars enter a freeway system at a rate of 100 cars/minute according to some unknown distribution, and the average time a car is in the freeway system is 22 mins. On average how many cars are either in the entry ramp or on the road?

(ii) [3] On average a car waits for 2mins in the ramp to enter the freeway. How many cars on average are waiting on the ramp?

(d) [6] $X_1, X_2, ...$ are i.i.d. Bernouli random variables. $P(X_i = 1) = \delta$. Let $S_n = X_1 + X_2 + ... + X_n$, and let m be some fixed positive integer. What is

$$\lim_{n\to\infty} \sum_{i=n\delta-m}^{n\delta+m} P(S_n = i)$$

Explain your answer.

Problem 2[20]

Bob gets three kinds of email: Critical, Not Critical and Spam. Each of these emails arrives as independent Poisson Processes with rates: λ_c, λ_{nc} and λ_s per hour respectively.

Bob's email spam filter classifies incoming emails and places each into one of two folders: Inbox and Spam. The filter attempts to place all emails that are Critical and Not Critical in the Inbox folder and to place the Spam emails in the Spam folder. Let p_c be the probability that the filter classifies a critical email correctly and places it in the inbox folder and $1 - p_c$ be the probability that it is placed in the spam folder. Similarly let p_{nc} be the probability that a non-critical email is placed in the inbox, and p_s be the probability that a spam email is placed in the spam folder.

(a) [10] Suppose that at some point in time that there are 3 emails in the Inbox folder. What is the probability that all three are Spam?

(b) [10] The probability that Bob will forward a Critical email in his Inbox to his colleagues is p_{cf}, and the probability that he will forward a Not Critical email in his Inbox to his colleagues is p_{ncf}. On average, how many emails does Bob forward in 1 hour?

Problem 3 [25]

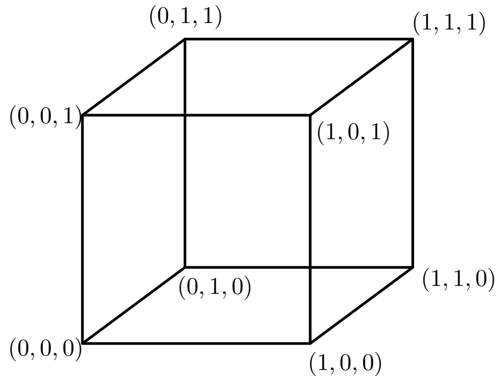

Figure 1: Unit cube

Lazy random walk on a cube: Consider a 3-dimensional unit cube as shown in Figure 1. An ant moves around on the vertices of this cube as follows: if at time n the ant is at a vertex v, then at time $n+1$ it remains on the same vertex v with probability $\frac{1}{2}$, or it jumps to one of the adjacent vertices with equal probability. Let X_n denote the position of the ant at time n. Note that X_n is a vector in $\mathbb{R}^3$.

(a) [5] Write the transition probability matrix P, and find the stationary distribution π.

(b) [7] Find the expected time to return to vertex $(0,0,0)$ for the first time, given that $X_0 = (0,0,0)$.

(c) [7] Find the expected time to reach $(1,1,1)$ for the first time, given that $X_0 = (0,0,0)$.

(d) [6] For $n \geq 1$, let $Y_n = X_n - X_{n-1}$. Thus Y_n is the direction the ant moved in at time n. Is $Y_1, Y_2, \ldots$ a Markov chain?

Problem 4 [30] Students arrive at a pastry shop according to a Poisson Process of rate $\lambda = 30$ per hour. The students independently buy Donuts or Eclairs, each with probability $\frac{1}{2}$. Assume that the shop opens at time zero and continues to be open indefinitely.

(a) [6] Conditional on 100 customers arriving in the first 5 hours of the shop opening, what is the probability that n donuts were sold in the first 4 hours of opening?

(b) [8] Find the pmf of the number of eclairs sold just before the first donut is sold.

(c) [8] Define the n^{th} sale reversal if the n^{th} customer buys something different from the $n-1$st customer. For example, in the sequence of sales $DDEDDEE$ the third fourth, and sixth sales are reversals. Now consider the process defined by the times of reversals. Find the expected time to the first reversal. Find the expected time between any two subsequent reversals.

(d) [8] Find the probability density function of the time between reversals.