EECS 20N: Structure and Interpretation of Signals and Systems   MIDTERM 3
Department of Electrical Engineering and Computer Sciences   21 November 2006
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name ___Eye___   FIRST Name ___Ell T.___

Lab Time ___365/24/60/60,___

- **(10 Points)** Print your name and lab time in legible, block lettering above (5 points) AND on the last page (5 points) where the grading table appears.

- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.

- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except three double-sided 8.5" × 11" sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff— *including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.

- **The exam printout consists of pages numbered 1 through 10.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the ten numbered pages. If you find a defect in your copy, notify the staff immediately.

- Please write neatly and legibly, because *if we can't read it, we can't grade it.*

- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*

- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.

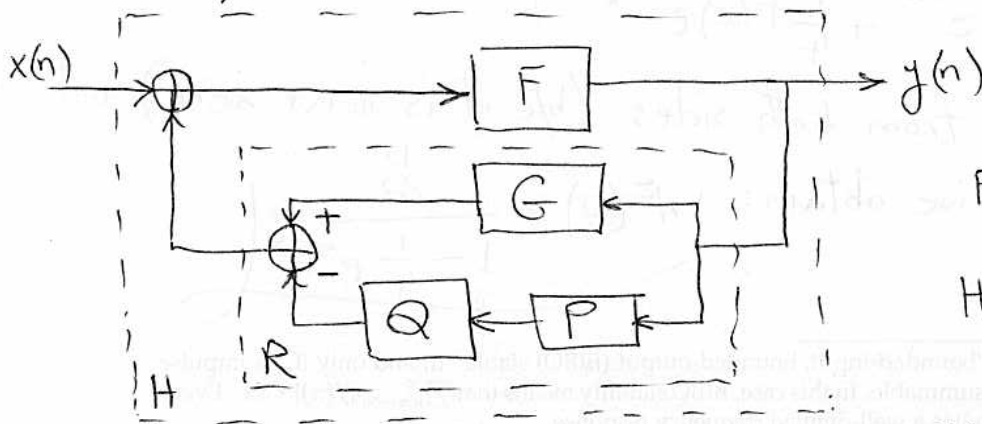- We hope you do a *fantastic* job on this exam.

You may use this page for scratch work only.
Without exception, subject matter on this page will *not* be graded.

MT3.1 (15 Points) Consider a well-structured interconnection H of discrete-time LTI systems F, G, P, and Q, as shown in the figure below, where each of the individual systems is a function defined on $[\mathbb{Z} \to \mathbb{C}] \to [\mathbb{Z} \to \mathbb{C}]$.
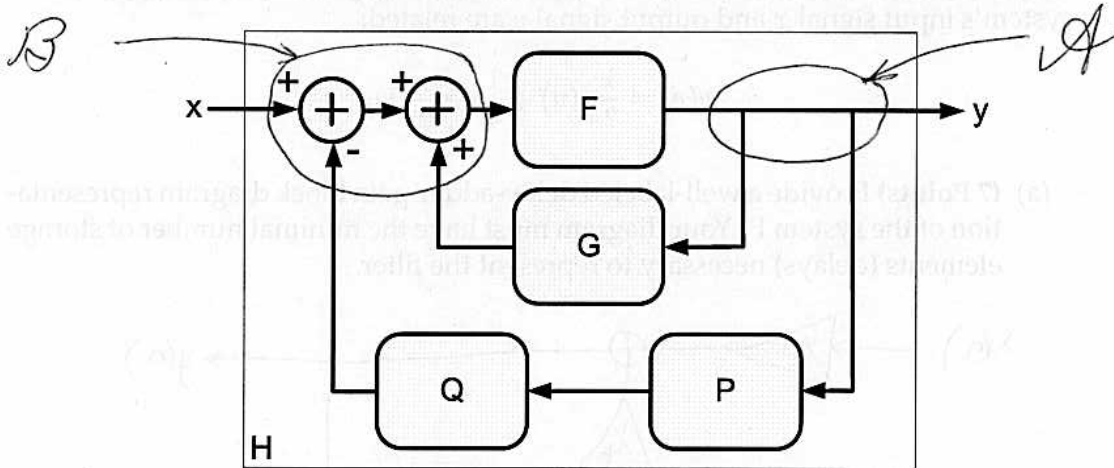
Let the respective frequency responses be F, G, P, and Q, defined on $\mathbb{R} \to \mathbb{C}$.

Determine the composite system's frequency response $H : \mathbb{R} \to \mathbb{C}$ in terms of the frequency responses of the components F, G, P, and Q. Reduce your expression to the simplest form possible.

We can redraw the figure as shown below, noting that $\mathcal{A}$ constitutes a single "node" (i.e., the same signal is fed to G and P), and that the adders of $\mathcal{B}$ can be slightly reorganized.
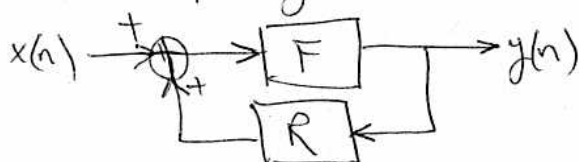

$$R(\omega) = G(\omega) - P(\omega)\,Q(\omega)$$

$$H(\omega) = \frac{F(\omega)}{1 - F(\omega)\,R(\omega)}$$

$$H(\omega) = \frac{F(\omega)}{1 - F(\omega)\big[G(\omega) - P(\omega)Q(\omega)\big]}$$

More compactly,

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MT3.2 (45 Points) A *causal, BIBO stable*[1] LTI filter F : $[\mathbb{Z} \to \mathbb{C}] \to [\mathbb{Z} \to \mathbb{C}]$ has impulse response $f : \mathbb{Z} \to \mathbb{R}$ and frequency response $F : \mathbb{R} \to \mathbb{C}$. The following linear, constant-coefficient difference equation at least partially describes how the system's input signal x and output signal y are related:

$$y(n) = \frac{1}{2} x(n) + \frac{1}{4} y(n-1).$$

(a) **(7 Points)** Provide a well-labeled delay-adder-gain block diagram representation of the system F. Your diagram must have the minimal number of storage elements (delays) necessary to represent the filter.

or

There is no unique answer.
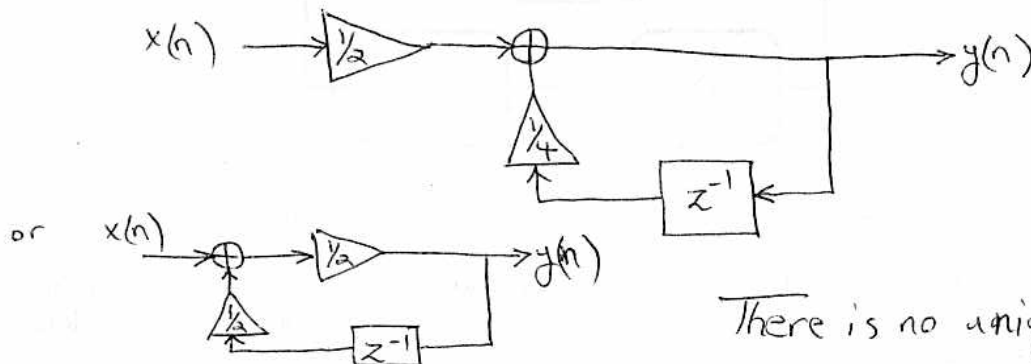
(b) **(8 Points)** Determine the filter's frequency response F by finding an expression for $F(\omega)$, $\forall \omega \in \mathbb{R}$.

$$x(n) = e^{i\omega n} \quad , \quad y(n) = F(\omega) e^{i\omega n} \quad \Longrightarrow \text{Plug into the LCCDE}$$

$$F(\omega) e^{i\omega n} = \frac{1}{2} e^{i\omega n} + \frac{1}{4} F(\omega) e^{i\omega(n-1)}$$

Removing $e^{i\omega n}$ from both sides (b/c it is never zero) and collecting terms, we obtain:

$$F(\omega) = \frac{\frac{1}{2}}{1 - \frac{1}{4} e^{-i\omega}}$$

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[1]An LTI system is "bounded-input, bounded-output (BIBO) stable" if, and only if, its impulse response is absolutely summable. In this case, BIBO stability means that $\sum_{n=-\infty}^{\infty} |f(n)| < \infty$. Every BIBO stable LTI system has a well-defined frequency response.
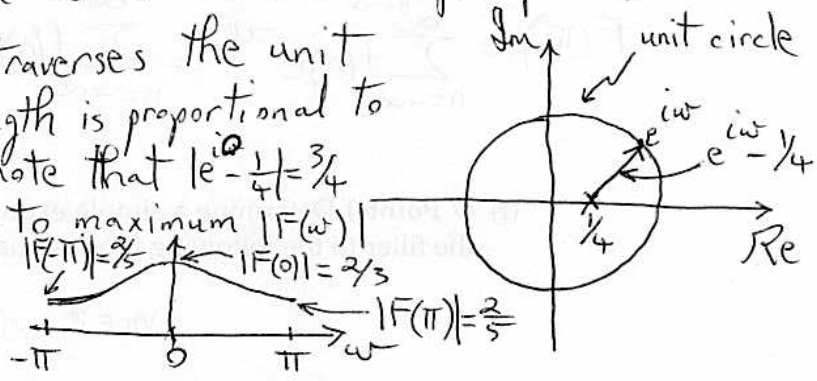
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(c) **(7 Points)** Provide a well-labeled sketch of the magnitude response values $|F(\omega)|$ over the frequency interval $-\pi \le \omega < \pi$.

Explain whether the filter is low-pass, band-pass, high-pass, all-pass, or none of these types. Provide a one-sentence explanation (not exceeding two lines) to support your choice.

Rewrite $F(\omega)$ as follows: $F(\omega) = \dfrac{\frac{1}{2}e^{i\omega}}{e^{i\omega} - \frac{1}{4}}$ $\Longrightarrow$

$|F(\omega)| = \dfrac{\frac{1}{2}}{\left|e^{i\omega} - \frac{1}{4}\right|}$. Use the graphical technique to sketch

$|F(\omega)|$. Treat $e^{i\omega} - \frac{1}{4}$ as a vector in the complex plane, and track its movement as $e^{i\omega}$ traverses the unit circle. The inverse of its length is proportional to $|F(\omega)|$ at any given ω. We note that $|e^{i0} - \frac{1}{4}| = \frac{3}{4}$ and $|e^{i\pi} - \frac{1}{4}| = \frac{5}{4}$, corresponding to maximum and minimum $|F(\omega)|$, respectively. F is a low-pass filter.

$|F(\pi)| = \frac{2}{5}$ $\quad$ $|F(0)| = 2/3$

(d) **(8 Points)** Determine a simple expression for the filter's impulse response f. That is, find an expression for $f(n)$, $\forall n \in \mathbb{Z}$. It is possible for you to solve this part independently of part (b).

You should be familiar with the time domain-frequency domain pair $\quad \alpha^n u(n) \longleftrightarrow \dfrac{1}{1 - \alpha e^{-i\omega}}$, if $0 < |\alpha| < 1$

Hence, you can write down the impulse response as

$$f(n) = \frac{1}{2}\left(\frac{1}{4}\right)^n u(n)$$

Alternatively, you can use the LCCDE to find $f(n)$. You're told the system is causal $\Longrightarrow f(n) = 0$, $\forall n < 0$. Now use the recursion:

$f(n) = \frac{1}{2}\delta(n) + \frac{1}{4}f(n-1)$

$f(0) = \frac{1}{2}\delta(0) + 0 = \frac{1}{2}$

$f(1) = 0 + \frac{1}{4}f(0) = \frac{1}{2}\left(\frac{1}{4}\right)$

5 $\quad f(2) = \frac{1}{4}f(1) = \frac{1}{2}\left(\frac{1}{4}\right)^2$

$\vdots$

$f(n) = \frac{1}{2}\left(\frac{1}{4}\right)^n$ $\quad n \ge 0$

(e) **(8 Points)** For each summation below, either determine a numerical value to which it converges, or explain why it does not converge.

(i) $\displaystyle\sum_{n=-\infty}^{\infty} f(n) =$

$$F(\omega) = \sum_{n=-\infty}^{\infty} f(n) e^{-i\omega n} \implies F(0) = \sum_{n=-\infty}^{\infty} f(n) = \frac{1/2}{1 - 1/4} = \frac{2}{3}$$

You can also use $\displaystyle F(0) = \sum_{n=-\infty}^{\infty} f(n) = \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{2}\cdot\frac{1}{1 - 1/4} = \frac{2}{3}$

(ii) $\displaystyle\sum_{n=-\infty}^{\infty} (-1)^n f(n) =$

$$F(\pi) = \sum_{n=-\infty}^{\infty} f(n) e^{-i\pi n} = \sum_{n=-\infty}^{\infty} f(n)(-1)^n = \frac{2}{5}$$

(f) **(7 Points)** Determine a simple expression for $y(n)$, $\forall n \in \mathbb{Z}$, the response of the filter to the following input signal x:

$$\forall n \in \mathbb{Z}, \quad x(n) = \cos\left(\frac{\pi}{3}n\right).$$

Hint: Use a straightforward geometric analysis. In carrying out your work, you may find the following approximations useful: $2/\sqrt{13} \approx 0.55$; $\pi/3 \approx 1.04$; and $\tan^{-1} 2\sqrt{3} \approx 1.29$.

We know $y(n) = |F(\pi/3)| \cos\left(\frac{\pi}{3}n + \angle F(\pi/3)\right)$. We must find $|F(\pi/3)|$ and $\angle F(\pi/3)$. Using a geometric analysis, we have the following picture at $\omega = \pi/3$.

$$\left|e^{i\pi/3} - \frac{1}{4}\right| = \sqrt{\left(\frac{1}{2}-\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{16} + \frac{3}{4}} = \frac{\sqrt{13}}{4}$$
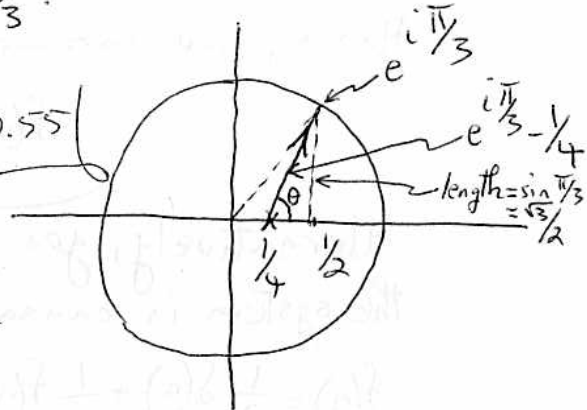
$$|F(\tfrac{\pi}{3})| = \frac{1/2}{|e^{i\pi/3} - 1/4|} = \frac{1/2}{\frac{\sqrt{13}}{4}} = \frac{2}{\sqrt{13}} \implies |F(\tfrac{\pi}{3})| = \frac{2}{\sqrt{13}} \approx 0.55$$

$$\angle F(\tfrac{\pi}{3}) = \angle\left(\frac{1}{2}e^{i\pi/3}\right) - \angle\left(e^{i\pi/3} - \frac{1}{4}\right)$$

$$\angle F(\tfrac{\pi}{3}) = \frac{\pi}{3} - \theta$$

θ on the diagram

$$\theta = \tan^{-1}\frac{\sqrt{3}/2}{\frac{1}{2}-\frac{1}{4}} = \tan^{-1}\frac{\sqrt{3}/2}{1/4} = \tan^{-1} 2\sqrt{3} \approx 1.29$$

$$\implies \angle F(\tfrac{\pi}{3}) = \frac{\pi}{3} - \tan^{-1} 2\sqrt{3} \approx 1.04 - 1.29 = -0.25$$

$$y(n) \approx 0.55 \cos\left(\frac{\pi}{3}n - \frac{1}{4}\right)$$

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MT3.3 (25 Points) A discrete-time LTI system $H : [\mathbb{Z} \to \mathbb{C}] \to [\mathbb{Z} \to \mathbb{C}]$ has impulse response $h : \mathbb{Z} \to \mathbb{R}$, where
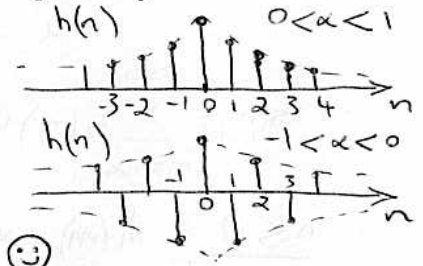
$$\forall n \in \mathbb{Z}, \quad h(n) = \alpha^{|n|},$$

$\alpha \in \mathbb{R}$, and $0 < |\alpha| < 1$.

(a) **(10 Points)** Determine the system's frequency response $H : \mathbb{R} \to \mathbb{C}$. Simplify your expression for $H(\omega)$, $\forall \omega \in \mathbb{R}$, to the extent possible. You need *not* plot the magnitude and phase responses $|H(\omega)|$ and $\angle H(\omega)$, respectively.

$$h(n) = \underbrace{\alpha^n u(n)}_{f(n)} + \underbrace{\alpha^{-n} u(-n)}_{g(n)} - \delta(n)$$

$$= f(n) + g(n) - \delta(n)$$

$\uparrow$ Why do we have to subtract this impulse? YOU DECIDE! ☺

$h(n)$ $0 < \alpha < 1$

$h(n)$ $-1 < \alpha < 0$

$$F(\omega) = \sum_{n=-\infty}^{\infty} f(n) e^{-i\omega n} = \frac{1}{1-\alpha e^{-i\omega}}$$

$$G(\omega) = \sum_{n=-\infty}^{\infty} f(-n) e^{-i\omega n} = \frac{1}{1-\alpha e^{i\omega}} \quad \leftarrow \text{See soln. to MT 3.4 (a)}.$$

$$H(\omega) = \frac{1}{1-\alpha e^{-i\omega}} + \frac{1}{1-\alpha e^{i\omega}} - 1 = \frac{1-\alpha e^{i\omega} + 1 - \alpha e^{-i\omega} - (1-\alpha e^{-i\omega})(1-\alpha e^{i\omega})}{(1-\alpha e^{-i\omega})(1-\alpha e^{i\omega})}$$

$$H(\omega) = \frac{2 - \alpha e^{i\omega} - \alpha e^{-i\omega} - \left(1 - \alpha e^{i\omega} - \alpha e^{-i\omega} + \alpha^2\right)}{1 - \alpha e^{i\omega} - \alpha e^{-i\omega} + \alpha^2} = \frac{1-\alpha^2}{1+\alpha^2 - 2\alpha \cos\omega} \quad \left(= \frac{1-\alpha^2}{(1-\alpha e^{-i\omega})(1-\alpha e^{i\omega})}\right)$$

You may use the blank space below for scratch work. Nothing written below this line on this page will be considered in evaluating your work.

Alternatively, note that h is an even function: $h(n) = h(-n)$, $\forall n \in \mathbb{Z}$.

$$H(\omega) = h(0) + \sum_{n=1}^{\infty} \left[h(n) e^{-i\omega n} + h(-n) e^{i\omega n} \right] = h(0) + \sum_{n=1}^{\infty} h(n) \left(e^{i\omega n} + e^{-i\omega n} \right)$$

$$H(\omega) = h(0) + \sum_{n=1}^{\infty} 2h(n) \cos(\omega n) \implies H(\omega) = 1 + 2\sum_{n=1}^{\infty} \alpha^n \cos \omega n$$
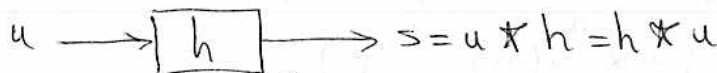
$\downarrow$ not in an enlightening form, but acceptable.

(b) **(10 Points)** Determine an expression for, and provide a well-labeled sketch of, the system's unit-step response[2] $s : \mathbb{Z} \to \mathbb{R}$.
Hint: Consider the cases $n \leq 0$ and $n > 0$ separately (or, alternatively, the cases $n < 0$ and $n \geq 0$ separately, whichever is easier for you).

Depending on how you solve this part, you may find one or both of the following identities useful.[3]

$$\sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}, \quad \text{if } 0 < |\alpha| < 1; \qquad \sum_{k=M}^{N} \alpha^k = \frac{\alpha^{N+1} - \alpha^M}{\alpha - 1}, \quad \text{if } \alpha \neq 1.$$

$$u \longrightarrow \boxed{h} \longrightarrow s = u * h = h * u$$

$$s(n) = \sum_{m=-\infty}^{\infty} h(m)\,u(n-m) = \sum_{m=-\infty}^{n} h(m) \quad \text{cumulative sum of } h.$$

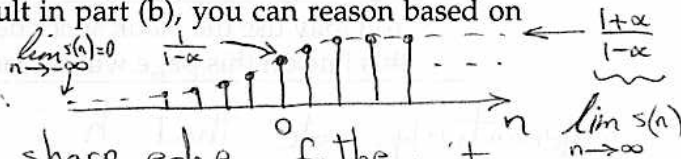$$\underline{n \leq 0} \qquad h(m) = \alpha^{-m} u(-m)$$

$$s(n) = \sum_{m=-\infty}^{n} h(m) = \sum_{m=-\infty}^{n} \alpha^{-m} = \alpha^{-n} + \alpha^{-(n-1)} + \alpha^{-(n-2)} + \cdots = \alpha^{-n} + \alpha^{-n+1} + \alpha^{-n+2} + \cdots$$

$$s(n) = \alpha^{-n}\left(1 + \alpha + \alpha^2 + \cdots\right) = \frac{\alpha^{-n}}{1-\alpha} \implies s(n) = \frac{\alpha^{-n}}{1-\alpha} \Bigg|_{n \leq 0}$$

$$\underline{n > 0} \qquad s(n) = \sum_{m=-\infty}^{n} h(m) = \underbrace{\sum_{m=-\infty}^{0} \alpha^{-m}}_{= \frac{1}{1-\alpha}} + \underbrace{\sum_{m=1}^{n} \alpha^{m}}_{\frac{\alpha^{n+1}-\alpha}{\alpha-1}} = \frac{1}{1-\alpha} - \frac{\alpha^{n+1}-\alpha}{1-\alpha} = \frac{1+\alpha - \alpha^{n+1}}{1-\alpha} \Bigg|_{n > 0}$$

(c) **(5 Points)** Viewed as a filter, is H low-pass, band-pass, high-pass, all-pass, or none of these? You need *not* look at the frequency response plots to make this determination. You need only look at the step response of part (b); however, if you are not confident of your result in part (b), you can reason based on the magnitude response plot. $\lim_{n\to\infty} s(n) = 0 \quad \frac{1}{1-\alpha}$

$\frac{1+\alpha}{1-\alpha} \longleftarrow$

$\lim_{n\to\infty} s(n)$

If $0 < \alpha < 1$, $s(n)$ will look like this. Clearly the filter has smoothened the sharp edge of the unit step input. So, in this case, the filter is low-pass.
If $-1 < \alpha < 0$, the filter will be high-pass.

[2]The unit-step response of a system is, as the name suggests, the response of the system to the unit-step input signal $u : \mathbb{Z} \to \mathbb{R}$, where $u(n) = 0$ if $n < 0$, and $u(n) = 1$ if $n \geq 0$.
[3]In the second summation, M and N are assumed to be finite.

MT3.4 (20 Points) Consider a discrete-time LTI system $F : [\mathbb{Z} \to \mathbb{C}] \to [\mathbb{Z} \to \mathbb{C}]$ that has impulse response f and frequency response $F : \mathbb{R} \to \mathbb{C}$.

(a) **(8 Points)** The impulse response g of a discrete-time LTI system $G : [\mathbb{Z} \to \mathbb{C}] \to [\mathbb{Z} \to \mathbb{C}]$ is defined as follows:

$$\forall n \in \mathbb{Z}, \quad g(n) = f(-n).$$

Explore how the frequency response G of system G relates to the frequency response F of system F; that is, derive a simple relation between $G(\omega)$ and $F(\omega)$, $\forall \omega \in \mathbb{R}$.

$$G(\omega) = \sum_{n=-\infty}^{\infty} g(n) e^{-i\omega n} = \sum_{n=-\infty}^{\infty} f(-n) e^{-i\omega n} = \sum_{m=-\infty}^{\infty} f(m) e^{i\omega m}$$

$$\text{Let } m = -n$$

$$= \sum_{m=-\infty}^{\infty} f(m) e^{-i(-\omega)m} = F(-\omega) \implies \underline{G(\omega) = F(-\omega)}$$

Note: The problem statement does <u>NOT</u> say that the impulse response f is a real-valued function. So it cannot be assumed that $F(-\omega) = F^*(\omega)$.

(b) **(12 Points)** The impulse response h of a discrete-time LTI system $H : [\mathbb{Z} \to \mathbb{C}] \to [\mathbb{Z} \to \mathbb{C}]$ is defined as follows:

$$\forall n \in \mathbb{Z}, \quad h(n) = e^{i\omega_0 n} f(n - N),$$

where ω_0 and N are constants such that $0 < \omega_0 < \pi$ and $N \in \mathbb{Z}$. Explore how the frequency response H of system H relates to the frequency response F of system F; that is, derive a relation between $H(\omega)$ and $F(\omega)$, $\forall \omega \in \mathbb{R}$.

Do this in two steps. Let $q(n) = f(n-N) \implies$

$$Q(\omega) = \sum_{n=-\infty}^{\infty} f(n-N) e^{-i\omega n} = \sum_{m=-\infty}^{\infty} f(m) e^{-i\omega(m+N)} = e^{-i\omega N} \underbrace{\sum_{m=-\infty}^{\infty} f(m) e^{-i\omega m}}_{F(\omega)}$$

$$m = n - N \implies n = m + N$$

Summary of Step 1: $\boxed{\underset{q(n)}{f(n-N)} \longleftrightarrow \underset{Q(\omega)}{e^{-i\omega N} F(\omega)}}$

Step 2: $h(n) = e^{i\omega_0 n} q(n) \implies H(\omega) = \sum_{n=-\infty}^{\infty} q(n) e^{i\omega_0 n} e^{-i\omega n} = \underbrace{\sum_{n=-\infty}^{\infty} q(n) e^{-i(\omega-\omega_0)n}}_{Q(\omega - \omega_0)}$

Frequency-shifted ↑ version of $Q(\omega)$.

Putting the two steps together: 9

$$\underline{H(\omega) = e^{-i(\omega-\omega_0)N} F(\omega - \omega_0)}$$

LAST Name ___Eye___ FIRST Name ___Ell T.___
Lab Time ___365/24/60/60___

Problem	Points	Your Score
Name	10	10
1	15	15
2	45	45
3	25	25
4	20	20
Total	**115**	115