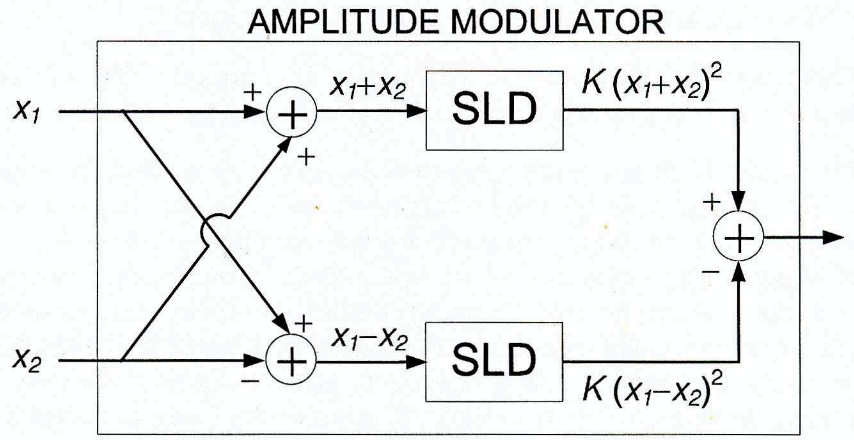
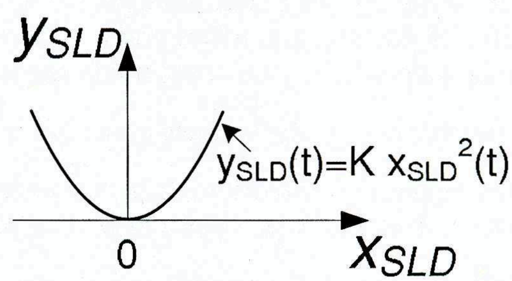
**MT2.1 (40 Points)** The block diagram below shows an architecture for implementing an amplitude modulator using signal adders and square-law devices (SLDs). The real-valued signals $x_1$ and $x_2$ are the inputs to the amplitude modulator and $y$ is its output.




Each square-law device is characterized by the following parabolic input-output graph, where $x_{\text{SLD}}$ denotes the input to the SLD and $y_{\text{SLD}}$ the output. The parameter $K$ is a positive constant.




**Formulas and Facts of Potential Use or Interest:**

$$\cos(\alpha+\beta) = \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta$$

$$\cos\alpha\,\cos\beta = \frac{1}{2}\Big[\cos(\alpha+\beta)+\cos(\alpha-\beta)\Big]$$

(a) Show that if $K = \frac{1}{4}$, the amplitude modulator output $y$ is characterized by

$$y(t) = x_1(t)\, x_2(t), \quad \forall t \in \mathbb{R}.$$

$$\left. \begin{array}{l} y = K\left[(x_1+x_2)^2 - (x_1-x_2)^2\right] = 4Kx_1x_2 \\ K = \frac{1}{4} \end{array} \right\} \Rightarrow \underline{y = x_1 x_2}$$

(b) Select the *strongest correct* assertion from the following choices. Explain your reasoning succinctly, but clearly and convincingly.

(I) The amplitude modulator above *must* be memoryless.
(II) The amplitude modulator above *can* be memoryless.
(III) The amplitude modulator above *cannot* be memoryless.

The system has no memory element. The SLD is memoryless because its input is instantaneously related to its output. Adders are memoryless. Also note that $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the input to the system $y(t) = f(x(t)) = x_1(t) x_2(t)$

(c) Select the *strongest correct* assertion from the following choices. Explain your reasoning succinctly, but clearly and convincingly.

(I) The amplitude modulator above *must* be causal.
(II) The amplitude modulator above *can* be causal.
(III) The amplitude modulator above *cannot* be causal.

Every memoryless system is causal.
(Do note, however, that the converse is not true)

(d) Select the *strongest correct* assertion from the following choices. Explain your reasoning succinctly, but clearly and convincingly.

(I) The amplitude modulator above *must* be time invariant.
(II) The amplitude modulator above *can* be time invariant.
(III) The amplitude modulator above *cannot* be time invariant.

Every memoryless system (as we define it in this course) is time invariant.



You can also argue it this way: $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \longrightarrow y(t) = x_1(t)\, x_2(t)$

Let $\hat{x}(t) = x(t-t_0) = \begin{bmatrix} x_1(t-t_0) \\ x_2(t-t_0) \end{bmatrix} \Rightarrow \hat{y}(t) = \hat{x}_1(t)\hat{x}_2(t) = x_1(t-t_0)x_2(t-t_0) = y(t-t_0)$

$\hat{x}(t) = \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix}$

(e) Select the *strongest correct* assertion from the following choices. Explain your reasoning succinctly, but clearly and convincingly.

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \longrightarrow \underbrace{\alpha \underline{x}(t)}_{\tilde{\underline{x}}(t)} = \begin{bmatrix} \alpha x_1(t) \\ \alpha x_2(t) \end{bmatrix}$$

(I) The amplitude modulator above *must* be linear.

(II) The amplitude modulator above *can* be linear.

$\boxed{\text{(III) The amplitude modulator above } cannot \text{ be linear.}}$

$$\tilde{\underline{x}}(t) \longrightarrow \text{System} \longrightarrow \tilde{y}(t) = \tilde{x}_1(t)\tilde{x}_2(t)$$

$$\tilde{y}(t) = \alpha^2 x_1(t) x_2(t) = \alpha^2 y(t)$$

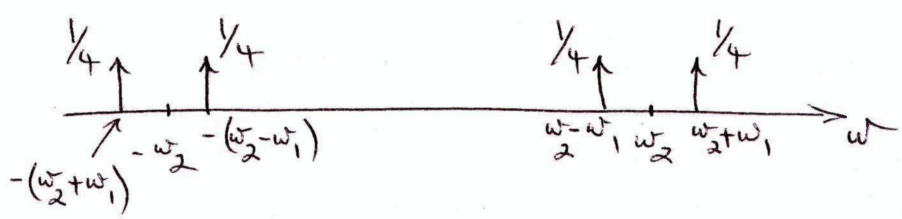
(f) Suppose the input signals $x_1$ and $x_2$ are sinusoids characterized by instantaneous values

$$x_1(t) = \cos(\omega_1 t) \quad \text{and} \quad x_2(t) = \cos(\omega_2 t), \quad \forall t,$$

where $0 < \omega_1 < \frac{\omega_2}{2}$. For simplicity, assume $K = \frac{1}{4}$.

(i) Provide a well-labeled sketch of the spectrum of the output signal $y$. Be sure to explain your reasoning succinctly, but clearly and convincingly.

$$y(t) = \cos \omega_1 t \cos \omega_2 t = \frac{1}{2}\left\{\cos\left[(\omega_2 + \omega_1)t\right] + \cos\left[(\omega_2 - \omega_1)t\right]\right\}$$




(ii) Prove that the output signal $y$ is periodic if, and only if, the ratio of the frequencies $\omega_1$ and $\omega_2$ is a rational number, i.e.,

$$\frac{\omega_1}{\omega_2} \in \mathbb{Q}.$$

$$y(t) = \frac{1}{2}\overbrace{\cos\left[(\omega_2+\omega_1)t\right]}^{\text{period } P_\alpha} + \frac{1}{2}\overbrace{\cos\left[(\omega_2-\omega_1)t\right]}^{\text{period } P_\beta}$$

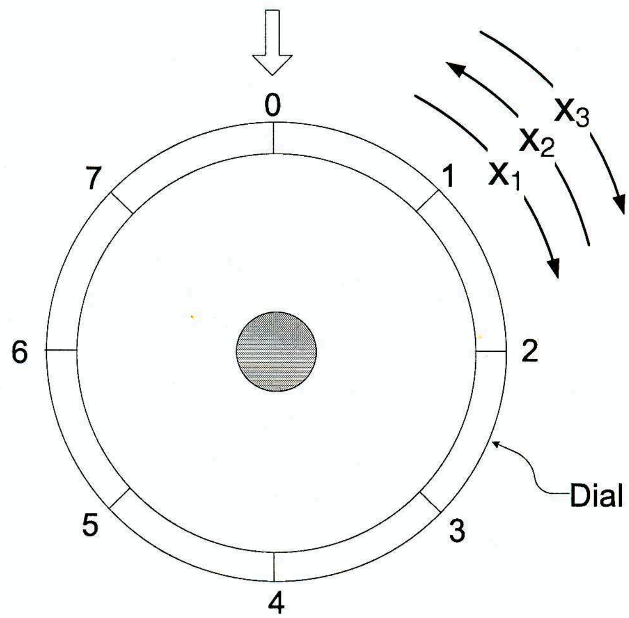
The sum of two periodic CT signals is period if, and only if, their respective fundamental periods are rational multiples of each other.

$$\left.\begin{aligned} P_\alpha &= \frac{2\pi}{\omega_2+\omega_1} \\ P_\beta &= \frac{2\pi}{\omega_2-\omega_1} \end{aligned}\right\} \Longrightarrow \text{Must have } \frac{P_\alpha}{P_\beta} = \frac{m}{n} \quad m, n \in \mathbb{N} \Longrightarrow$$

$$\frac{\frac{\cancel{2\pi}}{\omega_2+\omega_1}}{\frac{\cancel{2\pi}}{\omega_2-\omega_1}} = \frac{m}{n} \Longrightarrow \frac{\omega_2-\omega_1}{\omega_2+\omega_1} = \frac{m}{n} \Longrightarrow n\omega_2 - n\omega_1 = m\omega_2 + m\omega_1$$

$$\Longrightarrow (n-m)\omega_2 = (n+m)\omega_1 \Longrightarrow \frac{\omega_1}{\omega_2} = \frac{n-m}{n+m} \in \mathbb{Q}.$$

**MT2.2 (35 Points)** The following diagram shows a combination lock.




The lock can be opened only if an ordered sequence of *three* numbers—selected in strict adherence to the following steps—matches the lock's unique "combination":

**Step I.** Turn the dial clockwise two or more whole turns, and stop at the first number of the combination.

**Step II.** Turn the dial counter-clockwise one whole turn past the number in Step 1, and stop at the second number of the combination.

**Step III.** Turn the dial clockwise and stop at the third number of the combination.

The combination for this lock (i.e., the only sequence of numbers that opens it) is $(1, 3, 5)$. This means that to open the lock, a user must stop at 1 at the end of Step I; stop at 3 at the end of Step II; and stop at 5 at the end of Step III.

The thick vertical arrow at the top of the diagram is fixed, and it marks the number selected by the user at the end of every step.

The combination lock can be thought of as a mechanical system, where the "input signal" is the sequence of three numbers

$$x = (x_1, x_2, x_3) \in \{0, 1, 2, \ldots, 7\}^3$$

selected by the user who rotates the dial according to the steps and rules described above; needless to say, the sequence of numbers selected by the user may or may not match the lock's combination $(1, 3, 5)$.

The "output" signal $y = (y_1, y_2, y_3)$ shows the the sequence of *states* of the lock corresponding to the input sequence $(x_1, x_2, x_3)$. The state of the lock upon completion of each step described above is either *Locked* (L) or *Unlocked* (U).

By way of example, if the input signal is $(1, 2, 4)$, the corresponding output signal is (L, L, L), which means that the user has failed to open the lock. If, on the other hand, the user applies the input signal $(1, 3, 5)$, then the output signal will be (L, L, U), which means that the user has succeeded in opening the lock.

We can describe the combination lock by the function

$$\textsf{CombinationLock} : \quad \{0, 1, 2, \ldots, 7\}^3 \to \textsf{S}$$

$$y = \begin{cases} (\textsf{L}, \textsf{L}, \textsf{U}) & \text{if } x = (1, 3, 5) \\ (\textsf{L}, \textsf{L}, \textsf{L}) & \text{otherwise.} \end{cases}$$

The set S is something you will determine below.

In tackling this problem, consider only dial rotations that conform to the rules described above.

(a) Determine the size of the input signal space; that is, determine how many valid input signals $(x_1, x_2, x_3)$ exist.

$|\{0,1,\ldots,7\}^3| = 8^3$ Each $x_i$ has 8 possibilities. $(i=1,2,3)$

Also, determine S so that the description of the system is an *onto* function. What is the size of S?

There are only two possible sequences of states: (L, L, L) and (L, L, U).

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$S = \{(L,L,L), (L,L,U)\}$

(b) Select the *strongest correct* assertion from the following choices. Explain your reasoning succinctly, but clearly and convincingly.

(I) The combination lock *must* be a memoryless system.

(II) The combination lock *can* be a memoryless system.

(III) The combination lock *cannot* be a memoryless system.

Intuitively, we know that the lock must keep track of the sequence of the codes entered, so it must have memory.

More concretely, consider the following two input and output sequence pairs:

$$x = (1, 3, 5) \longrightarrow (L, L, U) = y$$

$$\hat{x} = (4, 2, 5) \longrightarrow (L, L, L) = \hat{y}$$

$x_3 = \hat{x}_3$ but $y_3 \neq \hat{y}_3$ so the system can't be memoryless.

(c) Select the *strongest correct* assertion from the following choices. Explain your reasoning succinctly, but clearly and convincingly.

(I) The combination lock *must* be a causal system.

(II) The combination lock *can* be a causal system.

(III) The combination lock *cannot* be a causal system.

- Every input signal pair $x$ and $\hat{x}$ that are identical up to, and including their respective first entries $x_1$ and $\hat{x}_1$, produce outputs $y$ and $\hat{y}$ that are identical up to, and including, their respective first entries $y_1$ and $\hat{y}_1$:

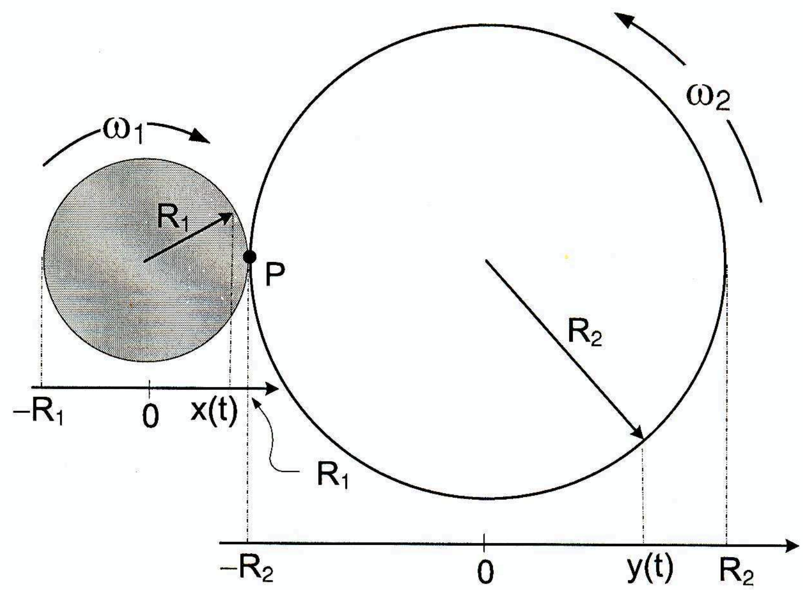
$$y = (\underset{\uparrow\, y_1}{L}, \star, \star) \qquad \hat{y} = (\underset{\uparrow\, \hat{y}_1}{L}, \star, \star)$$

- Every input signal pair $x$ and $\hat{x}$ that are identical up to, and including, their respective second entries $x_2$ and $\hat{x}_2$, produce outputs $y$ and $\hat{y}$ that are identical up to, and including, their respective second entries $y_2$ and $\hat{y}_2$:

$$y = (\underset{\uparrow\, y_1}{L}, \underset{\uparrow\, y_2}{L}, \star) \qquad \hat{y} = (\underset{\uparrow\, \hat{y}_1}{L}, \underset{\uparrow\, \hat{y}_2}{L}, \star)$$

- No pair of distinct inputs can be identical in all three entries $(x_1, x_2, x_3)$ $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$

**MT2.3 (30 Points)** Consider an ideal interlocking pair of rotating mechanical gears shown in the figure below. By ideal we mean that you can ignore friction and slippage between the gears.




The respective radii of the two gears are shown in the diagram, and are related according to $0 < R_1 < R_2$.

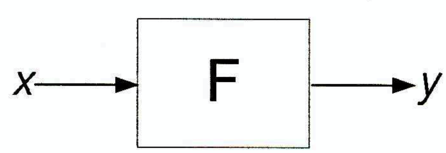
The smaller gear is the driver. The projection $x$ can be thought of as the input signal to this mechanical system.

The larger gear responds to the rotation of the smaller gear. Accordingly, we can think of the projection $y$ as the output signal of this mechanical system.

The smaller gear is shown rotating clockwise with angular velocity $\omega_1$ radians per second; the larger gear rotates in the opposite direction. At the contact point P where the two gears touch, they must have equal tangential velocities. Therefore,

$$\omega_1 R_1 = \omega_2 R_2.$$

An input-output depiction of the mechanical system is shown below.



The instantaneous values $x(t)$ and $y(t)$ of the input and output signals are shown in the figure and are described by

$$x(t) = R_1 \cos(\omega_1 t) \quad \text{and} \quad y(t) = R_2 \cos(\omega_2 t), \quad \forall t.$$

In answering the following questions, assume $\omega_1 = \pi/3$ and $R_1 = R_2/2$.

(a) Select the *strongest correct* assertion from the list below.

$x(t) = \frac{R_2}{2} \cos\left(\frac{\pi}{3} t\right)$

$y(t) = R_2 \cos\left(\frac{\pi}{6} t\right)$

- (i) F must be a time-invariant system.
- (ii) F can be a time-invariant system.
- $\boxed{\text{(iii) F cannot be a time-invariant system.}}$

Let $\hat{x}(t) = x(t-6) \implies \hat{x}(t) = \frac{R_2}{2} \cos\left(\frac{\pi}{3}(t-6)\right) = \frac{R_2}{2} \cos\left(\frac{\pi}{3} t\right) = x(t)$

If system were TI, then $\hat{y}(t) = y(t-6) = R_2 \cos\left(\frac{\pi}{6}(t-6)\right) = -R_2 \cos\left(\frac{\pi}{6} t\right)$

But since $\hat{x}$ is indistinguishable from $x$, then $\hat{y}$ must be identical to $y$. They're <u>not</u>!

(b) Select the *strongest correct* assertion from the list below.

- (i) F must be a memoryless system.
- (ii) F can be a memoryless system.
- $\boxed{\text{(iii) F cannot be a memoryless system.}}$

$x(0) = x(6) = \frac{R_2}{2}$

$y(0) = R_2 \neq -R_2 = y(6)$

Also can say

$\neg TI \implies \neg M$

(not TI) (not memoryless)

(c) Select the *strongest correct* assertion from the list below.

- $\boxed{\text{(i) F must be a causal system.}}$
- (ii) F can be a causal system.
- (iii) F cannot be a causal system.

Even if one of these two changes, you're looking @ a different system.

- The system is characterized by $R_1, R_2$. Once the input signal parameter $\omega_1$ is specified, the entire output is specified. The system need not peek ahead into the input signal to determine the output.
- You can look @ it this way as well: the only two input signals that are identical up to any point in time are $\cos \omega_1 t$ and $\cos(-\omega_1 t)$. These produce identical outputs.