EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 2
Department of Electrical Engineering and Computer Sciences 10 November 2009
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name Nautch FIRST Name Auntie

Lab Time ?

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except two double-sided $8.5\text{"} \times 11\text{"}$ sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

MT2.1 (65 Points) A discrete-time causal LTI system G has input x, corresponding output y, impulse response g, and frequency response G. Moreover, we can describe it by the linear constant-coefficient difference equation

$$y(n) = \alpha\, y(n-1) + \frac{1-\alpha}{2}\big[x(n) + x(n-1)\big], \quad \text{where} \quad |\alpha| < 1.$$

Note: You may or may not find it useful to know that if the impulse response f of an LTI system F is $f(n) = \alpha^n\, u(n)$, then its frequency response is $F(\omega) = \frac{1}{1-\alpha\, e^{-i\omega}}$.

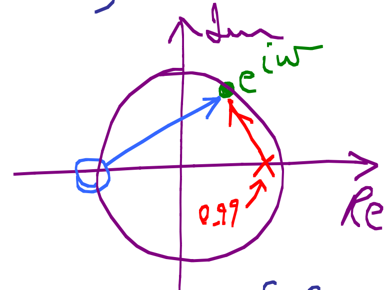
(a) Determine reasonably simple expressions for $G(\omega)$ and $g(n)$, the frequency response and impulse response values, respectively. Also, provide a well-labeled sketch of the magnitude response $|G(\omega)|$, if $\alpha = 0.99$. Use reasonable approximations in plotting $|G(\omega)|$, but be sure to explain your work.

Let $x(n) = e^{i\omega n} \Rightarrow y(n) = G(\omega)e^{i\omega n} \Rightarrow$ Applying to the LCCDE, we have:

$$G(\omega)\cancel{e^{i\omega n}} = \alpha e^{-i\omega} G(\omega)\cancel{e^{i\omega n}} + \frac{1-\alpha}{2}\cancel{e^{i\omega n}} + \frac{1-\alpha}{2} e^{-i\omega}\cancel{e^{i\omega n}} \Rightarrow \text{Collect terms}$$

and solve for $G(\omega)$:

$$G(\omega) = \frac{1-\alpha}{2}\,\frac{1+e^{-i\omega}}{1-\alpha e^{-i\omega}} = \frac{1-\alpha}{2}\,\frac{e^{i\omega}+1}{e^{i\omega}-\alpha}$$



$$|G(\omega)| = \frac{1-\alpha}{2}\,\frac{\|\nearrow\|}{\|\nearrow\|}$$

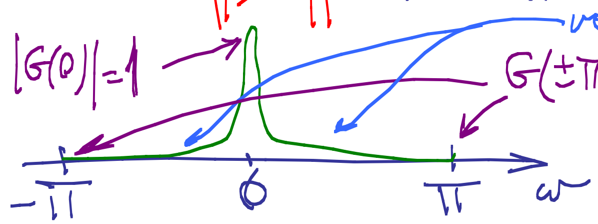
→ largest for $\omega = 0$

→ smallest for $\omega = \pi$

$|G(0)| = 1$

$G(\pm\pi) = 0$

very small values $\left(\sim \frac{1}{200}\right)$ in between $\omega = 0$ & $\omega = \pm\pi$.



$\boxed{\text{Anti-Notch Filter}}$

We know if $f(n) \to F(\omega)$ then $f(n-1) \to e^{-i\omega}F(\omega)$

So $\boxed{g(n) = \frac{1-\alpha}{2}\left[\alpha^n u(n) + \alpha^{n-1}u(n-1)\right]}$

(b) Make the strongest assertion you can about the BIBO stability of the system G. Explain your reasoning succinctly, but clearly and convincingly.

$$\sum |g(n)| = \frac{1-\alpha}{2}\sum_{n=0}^{\infty}\left|\alpha^n u(n) + \alpha^{n-1}u(n-1)\right| \le \frac{1-\alpha}{2}\left(\sum_{n=0}^{\infty}|\alpha|^n + \sum_{n=1}^{\infty}|\alpha|^{n-1}\right)$$

finite finite

b/c $|\alpha| < 1$

Hence, $\sum_n |g(n)| < \infty \Rightarrow$ G is BIBO stable

(c) A discrete-time causal LTI system H has input x, corresponding output y, impulse response h, and frequency response H given by

$$\forall \omega \in \mathbb{R}, \quad H(\omega) = \frac{1-\alpha}{2} \frac{1+e^{-iN\omega}}{1-\alpha e^{-iN\omega}},$$

where $N \in \{2, 3, 4, \ldots\}$, and α is the same as in system G.

(i) Determine the linear constant-coefficient difference equation that relates the input to the output of the system H.

We know that an LTI described by the LCCDE $\sum_{k=0}^{N} a_k y(n-k) = \sum_{\ell=0}^{M} b_\ell x(n-\ell)$ corresponds to the frequency response

$$H(\omega) = \frac{\sum_{\ell=0}^{M} b_\ell e^{-i\omega\ell}}{\sum_{k=0}^{N} a_k e^{-i\omega k}} \implies$$

Matching coefficients we have:

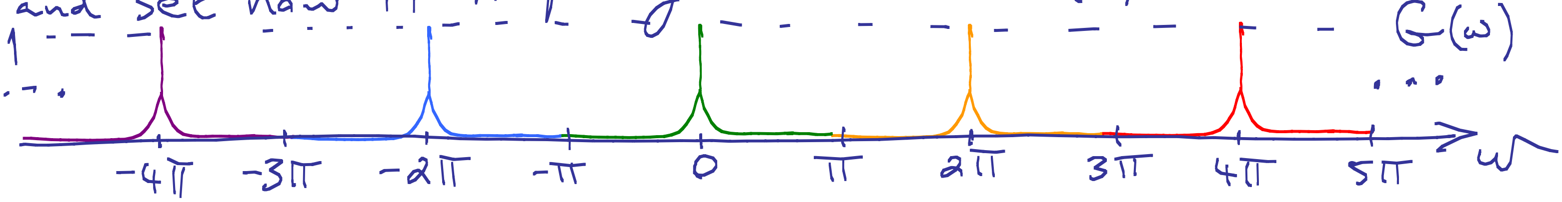
$$y(n) - \alpha y(n-N) = \frac{1-\alpha}{2}\left[x(n) + x(n-N)\right]$$

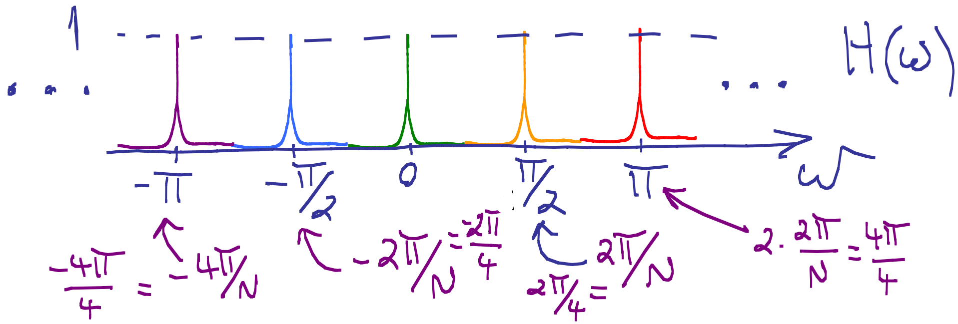
$$\implies \quad y(n) = \alpha y(n-N) + \frac{1-\alpha}{2}\left[x(n) + x(n-N)\right]$$

(ii) Provide a well-labeled sketch of the magnitude response $|H(\omega)|$ for the particular values $N = 4$ and $\alpha = 0.99$.

$H(\omega) = G(N\omega)$, $N = 4$ $\longrightarrow$ $H(\omega) = G(4\omega) \implies$

$H(\omega)$ is a frequency-contracted version of $G(\omega)$, by a factor of 4. We redraw $G(\omega)$ over a longer frequency interval and see how it frequency-contracts to $H(\omega)$:





like-colored anti-notches correspond to each other.

(iii) How does the impulse response h relate to the impulse response g?

$H(\omega) = G(N\omega) \implies$ h is obtained by upsampling g by a factor of N

$$h(n) = \begin{cases} g\left(\frac{n}{N}\right) & n \bmod N = 0 \\ 0 & \text{o/w} \end{cases}$$

$$g(n) \longrightarrow \boxed{\uparrow N} \longrightarrow h(n)$$

Verify: $H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n} = \sum_{\ell=-\infty}^{\infty} h(N\ell) e^{-i\omega N\ell}$

b/c $h(n) = 0$ for $n \bmod N \neq 0$

But $h(N\ell) = g(\ell) \implies H(\omega) = \sum_{\ell=-\infty}^{\infty} g(\ell) e^{-i(\omega N)\ell} = G(\omega N)$

(iv) Determine the output of the system H in response to the input signal

$$\forall n \in \mathbb{Z}, \quad x(n) = \sum_{k=0}^{2N-1} X_k \, e^{ik\pi n/N},$$

where the coefficients X_k are, in general, simply a set of complex-valued numbers.

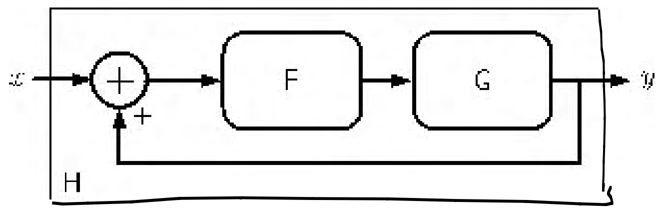
$$x(n) = X_0 + X_1 e^{i\frac{\pi}{N}n} + X_2 e^{i\frac{2\pi}{N}n} + \cdots + X_{2N-1} e^{i\frac{2N-1}{N}\pi n}$$

The filter H only passes even multiples of the frequency π/N and completely suppresses the odd multiples. The even multiples are passed w/o amplification or attenuation.

The output is

$$x(n) = X_0 + X_2 e^{i\frac{2\pi}{N}n} + \cdots + X_{2N-2} e^{i\frac{2N-2}{N}\pi n}$$

MT2.2 (20 Points) We form a system H by placing a pair of discrete-time LTI systems F and G in a feedback configuration, as shown below.



Let f, g, and h denote the impulse responses of the systems F, G, and H, respectively. Throughout this problem, you may assume that f, g, and h are absolutely summable; that is, $\sum_{n=-\infty}^{+\infty} |f(n)| < \infty$, $\sum_{n=-\infty}^{+\infty} |g(n)| < \infty$, and $\sum_{n=-\infty}^{+\infty} |h(n)| < \infty$.

Prove that

$$\frac{\sum_{n=-\infty}^{+\infty} h(n)}{1 + \sum_{n=-\infty}^{+\infty} h(n)} = \left(\sum_{n=-\infty}^{+\infty} f(n)\right)\left(\sum_{n=-\infty}^{+\infty} g(n)\right).$$

$$H(\omega) = \frac{F(\omega)G(\omega)}{1 - F(\omega)G(\omega)} \Rightarrow H(\omega) - F(\omega)G(\omega)H(\omega) = F(\omega)G(\omega) \Rightarrow$$

$$F(\omega)G(\omega) = \frac{H(\omega)}{1+H(\omega)} \xRightarrow[\omega=0]{\text{let}} F(0)G(0) = \frac{H(0)}{1+H(0)} \quad (a)$$

$$F(\omega) = \sum_{n=-\infty}^{\infty} f(n) e^{-i\omega n} \Rightarrow F(0) = \sum_{n=-\infty}^{\infty} f(n)$$

Similarly, $G(0) = \sum_{n=-\infty}^{\infty} g(n)$ and $H(0) = \sum_{n=-\infty}^{\infty} h(n)$

Applying these to Equation (a) we obtain the desired result:

$$\left(\sum_{n=-\infty}^{\infty} f(n)\right)\left(\sum_{n=-\infty}^{\infty} g(n)\right) = \frac{\sum_{n=-\infty}^{\infty} h(n)}{1 + \sum_{n=-\infty}^{\infty} h(n)}$$

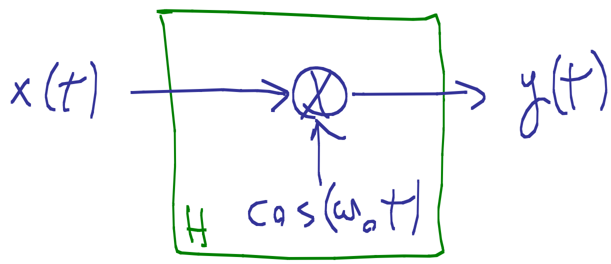
MT2.3 (20 Points) A continuous-time system is *anticausal* if for every $T \in \mathbb{R}$, whenever two inputs x_1 and x_2 are such that $x_1(\tau) = x_2(\tau)$ for all $\tau \geq T$, it follows that the corresponding outputs $y_1(\tau) = y_2(\tau)$ for all $\tau \geq T$. In short, the current output value does not depend on past input values.

True or False?

The intersection of the set of all causal systems and anticausal systems equals the set of memoryless systems.

Explain your reasoning fully. If you choose "false," then you must provide a system that is not memoryless, but which falls in the intersection described above.

The assertion is false.



$\omega_0 \neq 0$

The amplitude modulation system H is such that

$$y(t) = \cos(\omega_0 t)\, x(t)$$

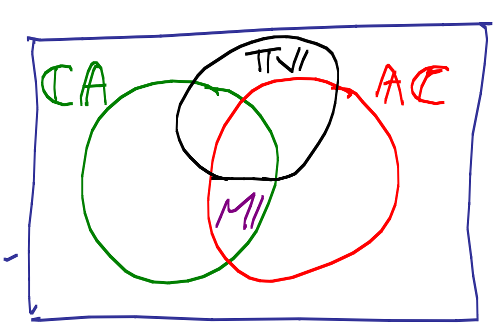
(an instantaneous, but not memoryless system)

This system is causal AND anti-causal, but because it is time-varying, it has memory.

In general, if M, CA, and AC denote the sets of memoryless, causal, and anti-causal systems, then $M \subset (CA \cap AC)$ a strict subset

strict

The set $(CA \cap AC) - M \subset TV$, where TV denotes the set of time-varying systems.



You may use this page for scratch work only.
Without exception, subject matter on this page will *not* be graded.

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Lab Time ?

Problem	Points	Your Score
Name	10	10
1	65	65
2	20	20
3	20	20
Total	**115**	115