EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 3
Department of Electrical Engineering and Computer Sciences 8 December 2009
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name Sambler FIRST Name Ubb

Lab Time ☺

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except three double-sided $8.5" \times 11"$ sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

Alternative Proof of Anti-Symmetry: $G(\omega) = \sum_{n=-\infty}^{\infty} g(n) e^{-i\omega n} \Rightarrow G(-\omega) = \sum_{n=-\infty}^{\infty} g(n) e^{i\omega n}$

Let $m = -n \Rightarrow G(-\omega) = \sum_{m=-\infty}^{\infty} g(-m) e^{-i\omega m}$. But $g(-m) = -g(m) \Rightarrow$

$G(-\omega) = -\sum_{m=-\infty}^{\infty} g(m) e^{-i\omega m} = -G(\omega) \Longrightarrow \boxed{G(-\omega) = -G(\omega)}$

MT3.1 (35 Points) A discrete-time LTI system G has impulse response g and frequency response G. Moreover, we know that the impulse response is real-valued and odd (anti-symmetric). That is,

$$g(n) \in \mathbb{R} \quad \text{and} \quad g(n) = -g(-n) \quad \text{for all } n.$$

(a) (20 Points) Prove that the frequency response $G(\omega)$ is

(i) odd (anti-symmetric) with respect ω; and ← See top of page for an independent alternative proof of anti-symmetry.

(ii) a purely imaginary function of ω.

$G(\omega) = \sum_{n=-\infty}^{\infty} g(n) e^{-i\omega n} = \sum_{n=-\infty}^{-1} g(n) e^{-i\omega n} + g(0) + \sum_{n=1}^{\infty} g(n) e^{-i\omega n}$

$g(n) = -g(-n) \Rightarrow g(0) = -g(0) \Rightarrow g(0) = 0$ characteristic of all odd functions.

$G(\omega) = \sum_{n=1}^{\infty} g(-n) e^{i\omega n} + \sum_{n=1}^{\infty} g(n) e^{-i\omega n} = -\sum_{n=1}^{\infty} g(n) e^{i\omega n} + \sum_{n=1}^{\infty} g(n) e^{-i\omega n} \Rightarrow$

$G(\omega) = -2i \sum_{n=1}^{\infty} g(n) \frac{e^{i\omega n} - e^{-i\omega n}}{2i} \Rightarrow \boxed{G(\omega) = -2i \sum_{n=1}^{\infty} g(n) \sin(\omega n)}$

We know $g(n) \in \mathbb{R}, \forall n \Rightarrow G(\omega)$ is purely imaginary. We've used the fact that $\sin(-\omega n) = -\sin(\omega n)$

Furthermore,

$G(-\omega) = -2i \sum_{n=1}^{\infty} g(n) \sin(-\omega n) = 2i \sum_{n=1}^{\infty} g(n) \sin(\omega n) = -G(\omega) \Rightarrow G$ is an odd function of ω.

(b) (15 Points) Determine the output y of the system G corresponding to an input signal having the following property:

$$x(n+2) = x(n) \quad \text{for all } n.$$

The signal x is periodic, and its period $p = 2$. Therefore, the fundamental frequency is at most $\omega_0 = \frac{2\pi}{2} = \pi \Longrightarrow$ the DFS expansion of x is: $x(n) = X_0 + X_1 e^{i\pi n}$

But $G(\omega)\Big|_{\substack{\omega = k\pi \\ k \in \mathbb{Z}}} = 0$, because $\sin(\omega n)\Big|_{\omega = k\pi} = \sin(2\pi k n) = 0$.

In particular, $G(0) = 0 = G(\pi)$. Therefore, output is $\boxed{y(n) = 0 \ \forall n}$

MT3.2 (70 Points) The figure below depicts a discrete-time causal LTI system G.

$$x \longrightarrow \boxed{\;G\;} \longrightarrow y$$

The frequency response of the system is expressed below:

$$\forall \omega \in \mathbb{R}, \quad G(\omega) = \frac{1 - e^{-i2\omega}}{1 - (0.99)^2 e^{-i2\omega}}.$$

(a) (10 Points) Provide a well-labeled plot of $|G(\omega)|$, the magnitude response of the filter. What type of filter is G? Explain.

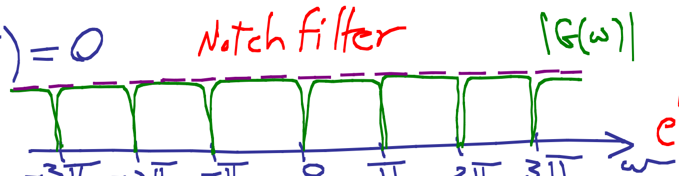
$$G(\omega) = \frac{e^{i2\omega} - 1}{e^{i2\omega} - (0.99)^2} = \frac{(e^{i\omega} - 1)(e^{i\omega} + 1)}{(e^{i\omega} - 0.99)(e^{i\omega} + 0.99)}.$$

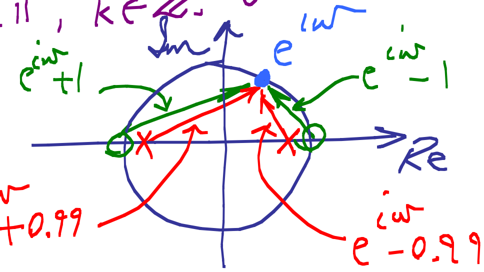
Note: $|e^{i\omega} - 1| \approx |e^{i\omega} - 0.99|$ and $|e^{i\omega} + 1| \approx |e^{i\omega} + 0.99|$ for all ω sufficiently away from $\omega = k\pi$, $k \in \mathbb{Z}$.

$|G(\omega)| \approx 1$ for ω not in a small neighborhood of $\omega = 0, \pi$.

$G(0) = G(\pi) = 0$

Notch filter





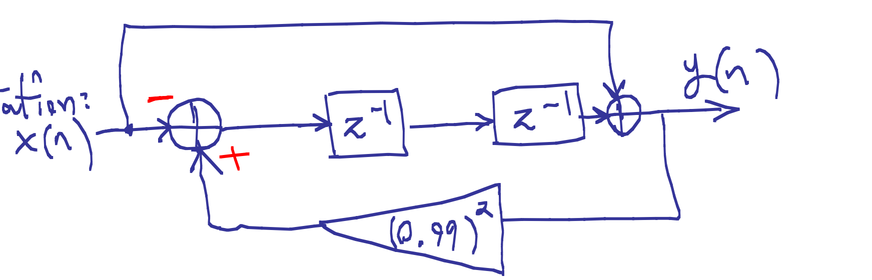
(b) (10 Points) Determine the linear constant-coefficient difference equation that that relates the input x to the output y.

$$\frac{Y(\omega)}{X(\omega)} = \frac{1 - e^{-i2\omega}}{1 - (0.99)^2 e^{-i2\omega}} \Rightarrow \left[1 - (0.99)^2 e^{-i2\omega}\right] Y(\omega) = \left(1 - e^{-i2\omega}\right) X(\omega) \Rightarrow$$

$$Y(\omega) - (0.99)^2 e^{-i2\omega} = X(\omega) - e^{-i2\omega} X(\omega) \overset{\mathcal{F}}{\Longleftrightarrow} y(n) - (0.99)^2 y(n-2) = x(n) - x(n-2)$$

To solve causally, rewrite and solve as $y(n) = (0.99)^2 y(n-2) + x(n) - x(n-2)$

A Delay-Adder-Gain block diagram of a causal implementation:



(c) (15 Points) Determine the impulse response g of the system.

Note that $G(\omega) = F(2\omega)$, where $F(\omega) = \dfrac{1-e^{-i\omega}}{1-(0.99)^2 e^{-i\omega}} \Longrightarrow$

$$g(n) = \begin{cases} f\left(\frac{n}{2}\right) & n \bmod 2 = 0 \text{ (i.e., n is even)} \\ 0 & n \bmod 2 \neq 0 \text{ (i.e., n is odd)} \end{cases}$$

That is, upsampling f by a factor of 2 yields g.

$$F(\omega) = \frac{1}{1-(0.99)^2 e^{-i\omega}} - \frac{e^{-i\omega}}{1-(0.99)^2 e^{-i\omega}} \Longrightarrow f(n) = (0.99)^{2n} u(n) - (0.99)^{2n-2} u(n-1)$$

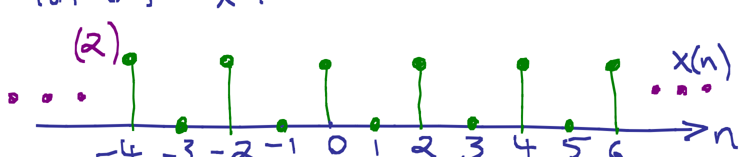
(d) (20 Points) Consider the following input signal:

$$x(n) = 1 + (-1)^n \quad \text{for all } n.$$

(i) If the input signal x is applied to the system G, determine the output signal y.

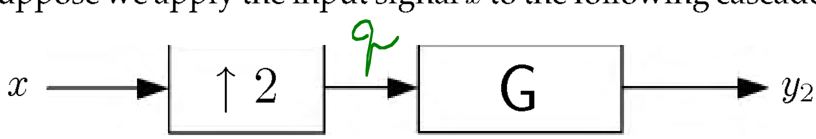
$$x(n) = e^{i0n} + e^{i\pi n} \Longrightarrow y(n) = G(0)e^{i0n} + G(\pi)e^{i\pi n} = 0 \text{ because } G(0) = G(\pi) = 0.$$

Plot of x:



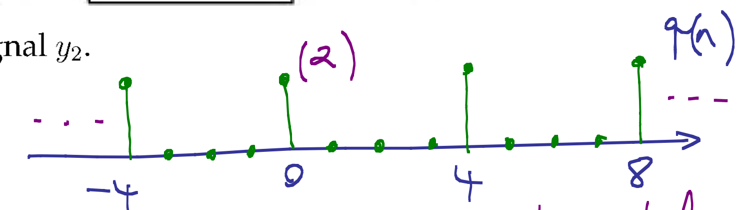
$$x(n) = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

(ii) Suppose we apply the input signal x to the following cascade of systems:



Determine the output signal y_2.

$$q(n) = \begin{cases} x(n/2) & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \Longrightarrow$$



The DFS expansion of q is

value of impulses

$$q(n) = \frac{2}{4} \sum_{k=-1}^{2} e^{ik\frac{\pi}{2}n} = \frac{1}{2}\left(e^{-i\frac{\pi}{2}n} + 1 + e^{i\frac{\pi}{2}n} + e^{i\pi n}\right)$$

P_q

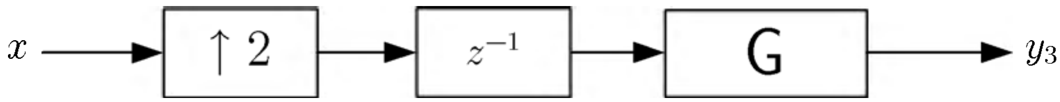
$P_q = 4$ fundamental period of q

$\omega_{0q} = \frac{2\pi}{4} = \frac{\pi}{2}$ fundamental frequency.

The filter G eliminates frequencies 0 and π, and passes the other two intact $\Longrightarrow y_2(n) = \dfrac{e^{i\frac{\pi}{2}n} + e^{-i\frac{\pi}{2}n}}{2}$

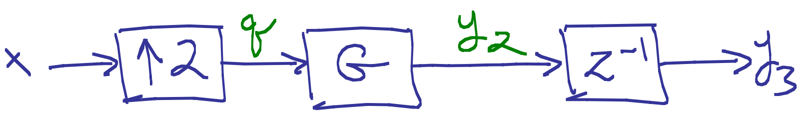
$\Longrightarrow$ $y_2(n) \approx \cos\left(\frac{\pi}{2}n\right)$

(iii) Suppose we apply the input signal x to the following cascade of systems:



Determine the output signal y_3.

We can reorder the filter and the unit-delay block because each is an LTI system:

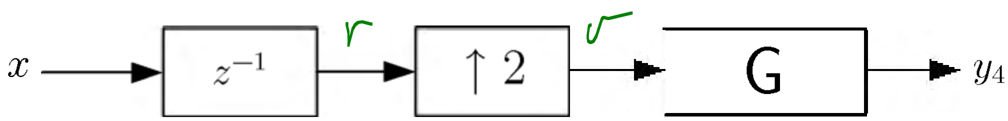


From this reordering, we note that $\boxed{y_3(n) = y_2(n-1)}$,

where y_2 is as defined in (d)(ii).

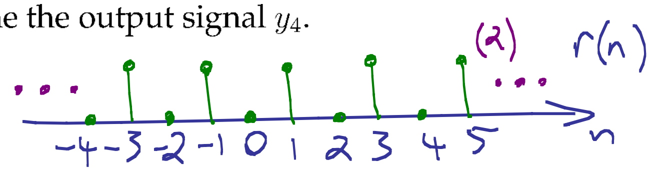
$y_3(n) = \cos\left(\frac{\pi}{2}(n-1)\right) = \cos\left(\frac{\pi}{2}n - \frac{\pi}{2}\right) \Longrightarrow \boxed{y_3(n) = \sin\left(\frac{\pi}{2}n\right)}$

(iv) Suppose we apply the input signal x to the following cascade of systems:

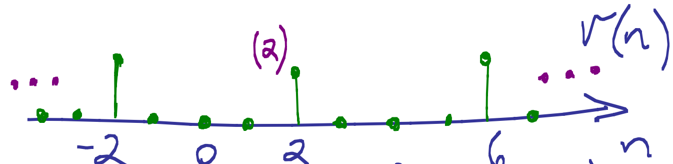


Determine the output signal y_4.

$r(n) = x(n-1) \Longrightarrow$



$v(n) = \begin{cases} r(n/2) & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \Longrightarrow$

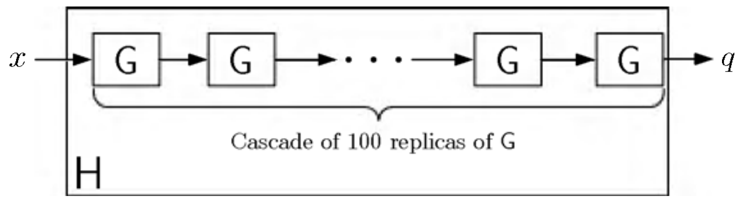


Note that the signal v is related to the signal q of (d)(ii) by a simple two-sample shift: $v(n) = q(n-2)$. Therefore,

$\boxed{y_4(n) = y_2(n-2)} \Rightarrow y_4(n) = \cos\left(\frac{\pi}{2}(n-2)\right) = \cos\left(\frac{\pi}{2}n - \pi\right) \Longrightarrow$

$\boxed{y_4(n) = -\cos\left(\frac{\pi}{2}n\right) = -y_2(n)}$

(e) (15 Points) Consider a system H constructed from the cascade of 100 replicas of the system G, as shown in the diagram below:



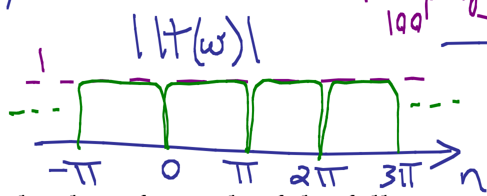
(i) Determine a reasonably simple expression for $H(\omega)$, the frequency response of the system H. Also, provide a well-labeled plot of $|H(\omega)|$, the magnitude response of the system H.

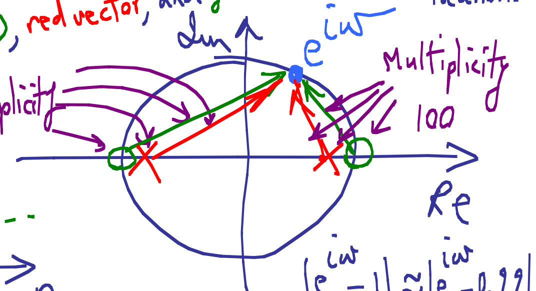
$H(\omega) = G^{100}(\omega) \Longrightarrow |H(\omega)| = |G(\omega)|^{100} \Longrightarrow$ Plot of $|H(\omega)|$ looks like that of $|G(\omega)|$, but with sharper notches.

$$H(\omega) = \frac{(e^{i\omega}-1)^{100}(e^{i\omega}+1)^{100}}{(e^{i\omega}-0.99)^{100}(e^{i\omega}+0.99)^{100}}$$

$\Longrightarrow$ The X-O diagram looks like G's, except there are 100 of each of X, O, red vector, and green vector at each marked location.

$$|H(\omega)| = \frac{|e^{i\omega}-1|^{100}\,|e^{i\omega}+1|^{100}}{|e^{i\omega}-0.99|^{100}\,|e^{i\omega}+0.99|^{100}}$$





$|e^{i\omega}-1| \approx |e^{i\omega}-0.99|$

$|e^{i\omega}+1| \approx |e^{i\omega}+0.99|$

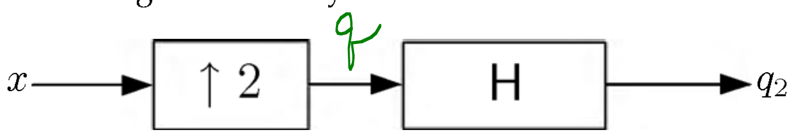
(ii) Determine numerical values for each of the following expressions:

$$\sum_{n=-\infty}^{+\infty} h(n) = \qquad \text{and} \qquad \sum_{n=-\infty}^{+\infty} (-1)^n h(n) =$$

$$\sum_{n=-\infty}^{\infty} h(n) = H(0) = G^{100}(0) = 0$$

$$\sum_{n=-\infty}^{\infty} (-1)^n h(n) = H(\pi) = G^{100}(\pi) = 0$$

(iii) Suppose we apply the input signal x described described in part (d) to the following cascade of systems:



Determine the output signal q_2.

$q_2(n) \approx y_2(n) \quad \forall n$

$H(0) = H(\pi) = 0$

$H(\frac{\pi}{2}) = H(-\frac{\pi}{2}) \approx 1$

LAST Name Sambler FIRST Name Ubb

Lab Time ☺

Problem	Points	Your Score
Name	10	10
1(a)	20	20
1(b)	15	15
2(a)	10	10
2(b)	10	10
2(c)	15	15
2(d)	20	20
2(e)	15	15
Total	**115**	115