MT2.1A Solutions

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1 SOLUTION

True.

There are two different ways to show this:

1. Easiest: Directly use properties of BIBO systems.
2. Harder: Use the time domain characterization of BIBO systems that are LTI.

Approach 1 is by far the easiest. We directly use the definition of BIBO stability, and thus provide a proof that works even for non-LTI systems that are BIBO.

Approach 2 is more effort. We need to show a particular identity which some students had difficulty demonstrating.

Furthermore, there is a third "approach" that "feels" right, but actually doesn't quite work that roughly 20 of you tried:

(A) Attempt to characterize BIBO LTI systems using the frequency domain characterization.

Approach A is interesting and "feels" right, but actually doesn't work.

1.1 Approach 1

Let $|x(n)|$ be absolutely bounded by $B_1 < \infty$. Since F is BIBO stable, the signal $\mathsf{F}[x(n)]$ is absolutely bounded by some $B_2 < \infty$. [1] Since G is BIBO stable, the signal $\mathsf{G}[\mathsf{F}[x(n)]]$ is absolutely bounded by some $B_3 < \infty$.

1.2 Approach 2

Here we use the fact that we have LTI systems. The LTI system H is BIBO stable $\iff \sum_n |h(n)| < \infty$.

Note that it is NOT sufficient to just show that $|h(n)| < \infty$ for all n. Even when $h(n) = 1 \forall n$, $\sum_n |h(n)| = \infty$.

OK, so now we have to show that $\sum_n |h(n)| < \infty$. This requires slogging through some tedious algebra. Another big reason to use Approach 1 over this technique.

Anyway:

$$\begin{aligned}
\sum_n |h(n)| &= \sum_n |\sum_k f(k)g(n-k)| \\
&\leq \sum_n \sum_k |f(k)||g(n-k)| \\
&= \sum_k |f(k)| \sum_n |g(n-k)|
\end{aligned}$$

This last equality requires an exchange of the order of summation. We are allowed to do this in this case because we are adding up a bunch of non-negative numbers.

[1] this bound B_2 is a constant dependent on B_1 and F. But more concise to write B_2 than B_2^{F,B_1}.

Proceeding further:

$$\begin{aligned}
\sum_k |f(k)| \sum_n |g(n-k)| &= \sum_k |f(k)| \sum_{\tilde{n}} |g(\tilde{n})| \text{ (where } \tilde{n} := n-k) \\
&< \sum_k |f(k)| C_g \text{ (by LTI characterization of the BIBO stable system } \mathsf{G}) \\
&< C_f C_g \text{ (same logic again, but for } \mathsf{F}) \\
&< \infty
\end{aligned}$$

This completes the proof. Notice how much more effort and error-prone this approach is compared to 1.

1.3 Approach A (INCORRECT)

We know that for LTI systems, BIBO stability can be characterized completely by this absolute summability criterion. In other words, given an LTI H, H is BIBO stable $\iff \sum_n |h(n)| < \infty$.

Another way to characterize LTIs is through their frequency response.

Basically, we want to prove that:

$$|H(\omega)| \leq C < \infty \forall \omega \iff \mathsf{H} \text{ is a BIBO stable system} \tag{1}$$

Before we prove this identity, let us discuss why this result makes it very easy to solve this problem.

Suppose that the relationship (1) holds true. Then since H is the cascade of the system F and G, then $H(\omega) = F(\omega)G(\omega)$. Thus when (1) holds true, we necessarily have that

$$|H(\omega)| = |F(\omega)||G(\omega)| \leq C_F C_G < \infty$$

By again by (1), $H(\omega) \leq C := C_F C_G < \infty$ is equivalent to H being BIBO stable.

So all the remains is to prove that (1) is true.

Let us attempt (and fail) to prove the $\implies$ direction.

1.3.1 Forward direction ($\implies$)

Suppose that $|H(\omega)| \leq C < \infty \forall \omega$. $|H(0)| \leq C < \infty$. However

$$\begin{aligned}
|H(0)| = |\sum_n h(n) e^{i0n}| \\
??? \\
\sum_n |h(n)|
\end{aligned}$$

As you can see, this proof isn't going to go anywhere. Convergence of a series doesn't imply absolute convergence of the series.

1.3.2 Backwards direction

Suppose that H is a BIBO stable system.

Then:

$$\begin{aligned}
&\sum_n |h(n)| < C \\
\implies &\sum_n |h(n) e^{i\omega n}| < C \forall \omega \\
\implies &|\sum_n h(n) e^{i\omega n}| < C \forall \omega \\
\iff &|H(\omega)| < C \forall \omega
\end{aligned}$$

as desired.

2 GRADING SCHEME

1. 0 points for saying false.
2. 6 points for true.
3. 8 points for a proof that is far away from being close, but something non-trivial.

   Approach A falls in this class.
4. 11 points for a proof that is close.
5. 15 points for a perfect proof.

Biggest mistakes people made:

- Thinking that BIBO stability can be shown by demonstrating that $h(n) < \infty \forall n$.
- Handwaving non-trivial things.
- Wrongly applying triangle inequality/Cauchy Shwarz inequality (this problem doesn't even require either.)
- Stating without proof that $\sum_n |f(n)| < \infty$ and $\sum_n |g(n)| < \infty \implies \sum_n |h(n)| < \infty$.
- This incorrect frequency domain approach that attempts to use $|H(\omega)| < \infty$ to prove that H is BIBO stable.

1 Question

True or False? It's *impossible* to find a causal system F and a non-causal system G such that $(f * g)(n)$ could serve as the impulse response of a causal LTI system $\hat{H}$.
Note that if a causal $\hat{H}$ is found, it means that a causal equivalent to H exists. It does *not* mean that H is causal. As described, H cannot be causal because at least one of its components must peek into the future of its input.
If your answer is that the claim is true, prove the nonexistence of causal equivalent to H. If your answer is that the claim is false, find a causal F and a non-causal G such that $\hat{h} = f * g$ can be considered the impulse response of a causal system $\hat{H}$.

2 Solution

The following is a perfect solution:
Answer. False.
Solution. (Counter-example): Let F be the causal LTI system with impulse response $h(n) = 0$ and let G be any non-causal LTI system. Then $(f * g) = 0$ (the zero signal), which is the impulse response of the zero system (the memoryless LTI system that maps all inputs to the zero signal).

3 Grading Scheme

The following letter codes were used:

3.1 B: Bad notation (-5 points)

This code was applied to solutions with particuarly bad notation. For example, if F is a system and x is a signal, writing

$$F(n) = x(n-1)$$

would warrant a B code. Other examples are shown in Common Errors (http://inst.eecs.berkeley.edu/ ee20n-te/handouts/mt2_solution_1b_common_errors.pdf).

3.2 E: Poorly worded or bad explaination (-10 points)

This code was used for solutions that had an idea of how to approach the problem but had contradictory or false statements in the proof. For example, writing

> F is causal

and later stating that

> F is noncausal

with no intention of proving by contradiction and without mentioning a contradiction would receive an E or lower. This was sparingly used.

3.3 L: Logical Error (-3 points)

This code was used for a variety of minor logical mistakes, the majority being leaving out a step in the proof. For example, providing valid examples of F and G and simply stating that H is causal and LTI without proof or calculation of the impulse response warranted an L.

3.4 M: Minor Mistake (-2 points)

This code was used for minor arithmetic errors. For example, writing the convolution formula as

$$(f * g)(n) = \sum_{k=-\infty}^{\infty} f(n)g(n-k)$$

would earn an M.

3.5 N: Little to no explaination or major logical error (at most 1 point)

This code was used for many attempts at proving the statement false that did not work. Here are some examples:

1. Using an example with F and G that simply do not work:

   > Let F be the LTI system with impulse response $f(n) = u(n)$ and G be the LTI system with impulse response $g(n) = u(-n)$. Then $h(n) = (f * g)(n) = \delta(n)$.

   The calculation of h is incorrect (it ends up being infinite for all values, including time before $n = 0$).

2. Trying to prove by restating the question:

   > Becuase there exists a causal LTI system F and a non-causal LTI system G such that F cascaded with G results in a causal LTI system H, the statement is false.

   This does not prove anything; it merely restates what needs to be proved.

3.6 T: answered True (at most 0 points)

Any solution that tried to prove the statement true received at most zero points.

MT2.2a:

If we feed in $e^{i\omega n}$, we should get $H(\omega)e^{i\omega n}$ out.

Thus, we should have the following equation:

$$H(\omega)e^{i\omega n} = 2\alpha H(\omega)e^{i\omega n}e^{-i\omega} - \alpha^2 H(\omega)e^{i\omega n}e^{-i2\omega} + e^{i\omega n}$$

Solving this equation yields the desired result.

Rubric:

- 5 points for being on the right track (i.e. letting x(n) equals the complex exponential)
- 10 points for somehow messing up the algebra but ended up at the correct result

Question 2.b

There are three ways to approach this problem

1. Using the hint in the question, we see that $H(\omega) = \frac{1}{(1-\alpha e^{-i\omega})^2}$

   So if we define $\zeta = \alpha e^{-i\omega}$, we can write $H(\omega) = \frac{1}{(1-\alpha e^{-i\omega})^2} = \sum_{n=0}^{\infty}(n+1)\alpha^n e^{-i\omega n}$ (*)

   The reason we can do this is because $|\alpha e^{-i\omega}| = |\alpha| < 1$

   We also know that $H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-i\omega n}$ (**)

   Looking at the patterns of the two summations, we suspect that $h(n) = (n+1)\alpha^n$, but the summation in (*) starts at zero while the one in (**) starts from -∞, to compensate for this, we define that $h(n) = (n+1)\alpha^n u(n)$, where u(n) is the unit step function.

Rubric: If you missed the argument that $|\alpha e^{-i\omega}| = |\alpha| < 1$ you received -1 point

If you only made a note about causality, you received -1 point

If you missed u(n) or did not make a specific note that h(n) = 0 for n<0, you received -2 points

If you wrote δ(n) instead of u(n), you received -1 point

If you missed to put n as the power for α, you received -2 points

If you had a wrong choice for ζ, or included the sigma in your final answer, or ended up with a wrong formula for h(n), you received overall of +8 points.

2. Using the technique Babak used in lecture, we write out the expressions for h(0), h(1), h(2), h(3) etc. and try to find a pattern that fits those terms
   We know
   h(n) = 0 for n<0 causal
   h(0) = 1
   h(1) = 2α
   h(2) = $3\alpha^2$
   h(3) = $4\alpha^3$
   h(4) = $5\alpha^4$
   we can find the pattern that for every n ≥0, h(n) = (n+1) α^n

Rubric: If you found (n+1) α^n but said h(n) = $\sum_{n=0}^{\infty}(n+1)\alpha^n$, or any other wrong expression, you received +8 points

If you tried to find a pattern but started off with the wrong LCCDE, you received +2 points.

3. Using g(n) from part d, if you convolve g(n) with itself you will get h(n)
   This way is more complicated than the other two

   g(n) = α^nu(n)

   h(n) = $\sum_{k=-\infty}^{\infty} \alpha^k u(k)\alpha^{n-k}u(n-k)$, the unit step functions determine the bounds of the summation:
   u(k) is non zero for k≥0
   u(n-k) is non zero for k≤n
   so this summation will have a non zero value if n≥0, so then k can have a value between 0 and n and both unit steps will be one ➔ $0 \le k \le n$
   so for $n<0$ the summation is zero ➔ $h(n) = 0$ for $n<0$

   for n ≥ 0, the summation will go from 0 to n ➔ h(n) = $\sum_{k=0}^{n} \alpha^k \alpha^{n-k}$ =$\alpha^n \sum_{k=0}^{n} 1$ = (n+1) α^n

Rubric: If you tried convolving but had the wrong expression for g(n) from part d, you received +5 points.

MT 2.2c Solution

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Let $\alpha = \frac{1}{2}$. Determine the output of the filter H in response to the input $x(n) = 1 + (1)^n$.

Solution

This problem was meant to test whether you can recognize that an LTI system's response to a sum of different frequencies is a sum of each frequency term, scaled by each corresponding value of the frequency response. We first rewrite the signal in a form which makes its constituent frequencies more obvious:

$$x(n) = e^{i0n} + e^{i\pi n}$$

Recall that the frequency response was given to be

$$H(\omega) = \frac{1}{(1 - \alpha e^{-i\omega})^2}$$

Then the output of the system H with this signal fed in as the input is:

$$y(n) = H(\omega = 0)e^{i0n} + H(\omega = \pi)e^{i\pi n}$$

$$y(n) = \frac{1}{(1 - \frac{1}{2}e^{-i0})^2}e^{i0n} + \frac{1}{(1 - \frac{1}{2}e^{-i\pi})^2}e^{i\pi n}$$

$$y(n) = \frac{1}{(1 - \frac{1}{2}(1))^2}e^{i0n} + \frac{1}{(1 - \frac{1}{2}(-1))^2}e^{i\pi n}$$

$$y(n) = 4e^{i0n} + \frac{4}{9}e^{i\pi n}$$

$$y(n) = 4 + \frac{4}{9}(-1)^n$$

Note that we are *NOT* using $y(n) = H(\omega)x(n)$, which is incorrect!

Grading Scheme

- **+5** points for recognizing to write the signal as a sum of the frequencies $\omega = 0$ and $\omega = \pi$.
- **+4** points for writing the output $y(n)$ as the sum of these frequencies scaled by corresponding values of $H(\omega)$.
- **+1** point for correct solution
- Several students attempted to convolve the input signal with the impulse response found in part 2b. While this is a valid approach, it is less direct than the above solution and does not use what you have learned about frequency behavior. As such, this approach was treated mostly with an all-or-nothing grading scheme.

Question 2(d) Solution:

Determine $G(\omega)$ and $g(n)$ and provide a well-labeled approximate sketch of $|H(\omega)|$

Since the system H can be implemented as a cascade interconnection of two identical causal, BIBO stable LTI systems with those systems being G we know the following relationship:

$$H(\omega) = G(\omega)G(\omega) = G(\omega)^2$$

$$G(\omega) = \sqrt{H(\omega)}$$

$$\boxed{G(\omega) = \frac{1}{(1 - \alpha e^{-i\omega})}}$$

We now want to find $g(n)$. We know from $G(\omega)$ that the relationship between the impulse and frequency responses are as follows:

$$G(\omega) = \sum_{n=-\infty}^{+\infty} g(n)e^{-i\omega n}$$

From here we can see that $G(\omega)$ looks as follows:

$$G(\omega) = \frac{1}{1-r} \;, \qquad r = ae^{-i\omega}$$

This looks like a geometric series:

$$\sum_{n=0}^{+\infty} r^n = \frac{1}{1-r} \;, \qquad |r| < 1$$

We can use this property as long as we prove $|r| = \left|ae^{-i\omega}\right| < 1$

This is true since we are given that $|\alpha| < 1$ and we know the complex exponential $\left|e^{-i\omega}\right| = 1$ for all ω and therefore the product of the two must be less than 1 as well and therefore since $\left|\alpha e^{-i\omega}\right| < 1$ we can say the following:

$$G(\omega) = \frac{1}{1-r} = \sum_{n=0}^{+\infty} r^n = \sum_{n=0}^{+\infty} \left(\alpha e^{-i\omega}\right)^n$$

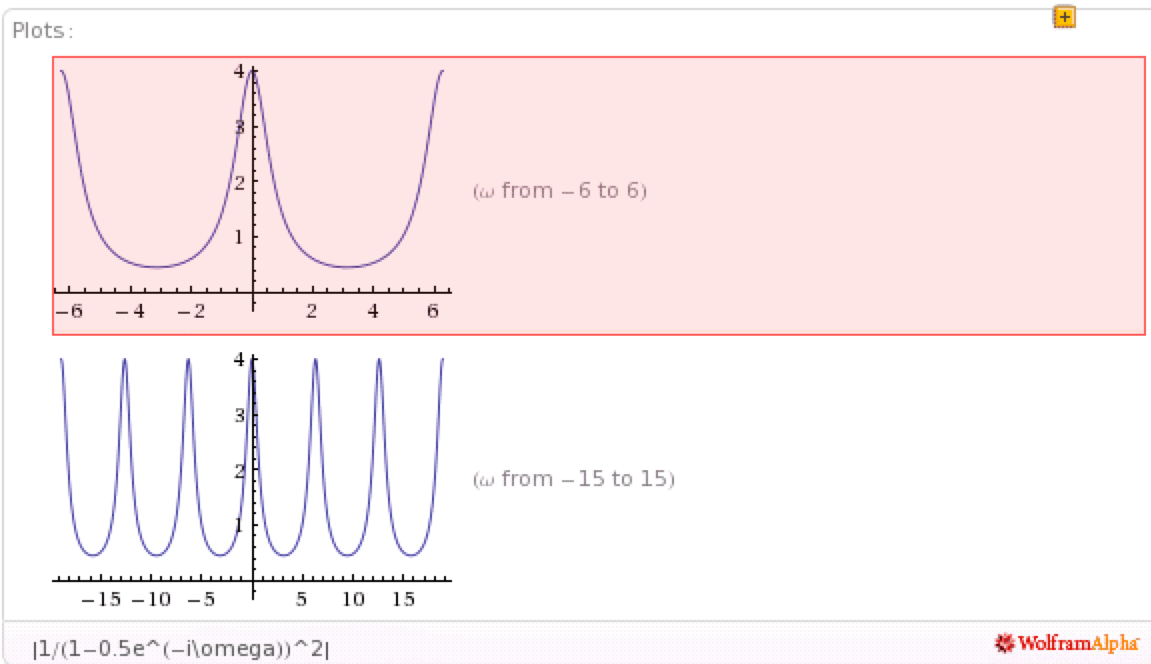
$$= \sum_{n=0}^{+\infty} \alpha^n e^{-i\omega n}$$

We need to be careful and note that the bottom bound of the summation is 0 and NOT negative infinity. Therefore in order to change the bounds to match we can multiply our system by the unit step.

$$\sum_{n=-\infty}^{+\infty} \alpha^n u(n) e^{-i\omega n} \qquad \textit{Pattern Match With} \qquad \sum_{n=-\infty}^{+\infty} g(n)e^{-i\omega n}$$

$$\boxed{g(n) = a^n u(n)}$$

$|H(\omega)|$



We were looking to make sure you had labels, ploted $|H(\omega)|$ (instead of $|G(\omega)|$), label for highest magnitude of 4 and lowest of $\frac{4}{9}$ at $\omega = \pm\pi$.

Question 2(d) Grading:

3 points total for finding $G(\omega)$ (+3 for fully correct solution of $G(\omega)$)

- +2 for incorrect steps in derivation but mostly correct

4 points total for finding $g(n)$ (+4 for fully correct solution of $g(n)$)

- +3 point if you did not state that $|\alpha e^{-iw}| < 1$ if you derived using geometric series but had correct $g(n)$
- +3 point if your derivation to $g(n)$ had illogical steps but was mostly correct.
- +2 points if you forgot $u(n)$ or did not state in any way that $g(n) = 0\,, n < 0$ but everything else was correct

3 points total for plotting $|H(\omega)|$ (+3 for fully correct graph)

- +2 for no labels of values but curve and other labels were generally correct
- +2 for error in graph such as incorrect values or incorrect values of ω but everything else generally correct.
- +0 for plotted $|G(\omega)|$ when you were asked to plot $|H(\omega)|$

MT2.3 Solutions

November 8, 2011

1 SOLUTION

1.1 Deriving $A(\omega)$

From the definition of $h(n)$, it is zero for n outside of the range $[0, M]$, and is symmetrical about $\frac{M}{2}$.

$$\begin{aligned}
H(\omega) &= \sum_{k=-\infty}^{\infty} h(k)e^{-i\omega k} \\
&= \sum_{k=0}^{M} h(k)e^{-i\omega k} \\
&= \sum_{k=0}^{\frac{M-1}{2}} h(k)e^{-i\omega k} + \sum_{k=\frac{M+1}{2}}^{M} h(k)e^{-i\omega k} \\
&\text{Let } n = M-k, \text{ so } k = M-n \\
&= \sum_{k=0}^{\frac{M-1}{2}} h(k)e^{-i\omega k} + \sum_{n=\frac{M-1}{2}}^{0} h(M-n)e^{-i\omega(M-n)} \\
&= \sum_{k=0}^{\frac{M-1}{2}} h(k)e^{-i\omega k} + \sum_{n=0}^{\frac{M-1}{2}} h(n)e^{i\omega n}e^{-i\omega M} \\
&= \sum_{n=0}^{\frac{M-1}{2}} h(n)\left(e^{-i\omega n} + e^{i\omega n}e^{-i\omega M}\right) \\
&= \sum_{n=0}^{\frac{M-1}{2}} e^{-i\omega\frac{M}{2}} h(n)\left(e^{-i\omega n}e^{i\omega\frac{M}{2}} + e^{i\omega n}e^{-i\omega\frac{M}{2}}\right) \\
&= e^{-i\omega\frac{M}{2}} \sum_{n=0}^{\frac{M-1}{2}} h(n)\left(e^{-i\omega\left(n-\frac{M}{2}\right)} + e^{i\omega\left(n-\frac{M}{2}\right)}\right) \\
&= e^{-i\omega\frac{M}{2}} \sum_{n=0}^{\frac{M-1}{2}} 2h(n)\cos\left(\omega\left(n-\frac{M}{2}\right)\right) \\
&= e^{-i\omega\frac{M}{2}} A(\omega)
\end{aligned}$$

Thus,

$$A(\omega) = \sum_{n=0}^{\frac{M-1}{2}} 2h(n)\cos\left(\omega\left(n-\frac{M}{2}\right)\right)$$

1.2 Showing $A(\omega)$ is odd about $\omega = \pi$

To show this, we need to show that $A(\pi - \Omega) = -A(\pi + \Omega)$ for all Ω. Let's consider:

$$A(\pi - \Omega) = \sum_{n=0}^{\frac{M-1}{2}} 2h(n)\cos\left((\pi - \Omega)\left(n - \frac{M}{2}\right)\right) \qquad -A(\pi + \Omega) = -\sum_{n=0}^{\frac{M-1}{2}} 2h(n)\cos\left((\pi + \Omega)\left(n - \frac{M}{2}\right)\right)$$

From looking at the above, we are done if we can show that $\cos\left((\pi - \Omega)\left(n - \frac{M}{2}\right)\right) = -\cos\left((\pi + \Omega)\left(n - \frac{M}{2}\right)\right)$:

$$\begin{aligned}
\cos\left((\pi - \Omega)\left(n - \frac{M}{2}\right)\right) &= \cos\left(-(\pi - \Omega)\left(n - \frac{M}{2}\right)\right) \\
&= \cos\left((\Omega - \pi)\left(n - \frac{M}{2}\right)\right) \\
&= \cos\left(\Omega\left(n - \frac{M}{2}\right) - \pi n + \pi\frac{M}{2}\right) \\
&\text{Adding an odd multiple of } \pi \text{ would negate the cosine...} \\
&= -\cos\left(\Omega\left(n - \frac{M}{2}\right) - \pi n + \pi\frac{M}{2} - \pi M\right) \\
&\text{...and adding an even multiple of } \pi \text{ would not} \\
&= -\cos\left(\Omega\left(n - \frac{M}{2}\right) - \pi n + \pi\frac{M}{2} - \pi M + \pi 2n\right) \\
&= -\cos\left(\Omega\left(n - \frac{M}{2}\right) + \pi n - \pi\frac{M}{2}\right) \\
&= -\cos\left(\Omega\left(n - \frac{M}{2}\right) + \pi\left(n - \frac{M}{2}\right)\right) \\
&= -\cos\left((\Omega + \pi)\left(n - \frac{M}{2}\right)\right)
\end{aligned}$$

1.3 Showing $A(\omega)$ is periodic with period 4π

To show this, we need to show that $A(\Omega + 4\pi) = A(\Omega)$ for all Ω. Let's consider:

$$A(\Omega + 4\pi) = \sum_{n=0}^{\frac{M-1}{2}} 2h(n)\cos\left((\Omega + 4\pi)\left(n - \frac{M}{2}\right)\right) \qquad A(\Omega) = \sum_{n=0}^{\frac{M-1}{2}} 2h(n)\cos\left(\Omega\left(n - \frac{M}{2}\right)\right)$$

Similar to above, all we need to show is that $\cos\left((\Omega + 4\pi)\left(n - \frac{M}{2}\right)\right) = \cos\left(\Omega\left(n - \frac{M}{2}\right)\right)$:

$$\begin{aligned}
\cos\left((\Omega + 4\pi)\left(n - \frac{M}{2}\right)\right) &= \cos\left(\Omega\left(n - \frac{M}{2}\right) + 4\pi\left(n - \frac{M}{2}\right)\right) \\
&= \cos\left(\Omega\left(n - \frac{M}{2}\right) + 4\pi n - 2M\pi\right) \\
&\text{Adding an even multiple of } \pi \text{ does not change the cosine} \\
&= \cos\left(\Omega\left(n - \frac{M}{2}\right)\right)
\end{aligned}$$

2 GRADING SCHEME

2.1 Basic Point Breakdown

- 20 points for the derivation of $A(\omega)$

  +2 points for changing the summation limits correctly

  +10 points for doing something along the lines of splitting the sum, with variable substitution

  +6 points for massaging the expression to get the cosine

  +2 points for general math

- 3 points for showing oddness about $\omega = \pi$

  +1 point for identifying what we need to show

  +2 points for work

- 2 points for showing 4π-periodicity

  +1 point for identifying what we need to show

  +1 point for work

2.2 Notes

- Some people didn't understand what the definition of $h(n)$ meant (since it included $h(n)$ itself). If you try out different $h(n)$'s for different values of M, it becomes clear that the definition was just supposed to mean that $h(n)$ is symmetrical about $\frac{M}{2}$.
- Many of you tried to reverse-engineer $A(\omega)$, showing that $A(\omega)e^{-i\omega\frac{M}{2}}$ is equal to $H(\omega)$. This is okay.
- Few people understood what oddness about $\omega = \pi$ meant. Most tried to show something like $A(\pi) = -A(-\pi)$.
- Some people argued that since $H(\omega)$ is periodic with period 2π, $A(\omega)$ is also periodic with period 2π, so it is also periodic with period 4π. This is not true. $A(\omega)$ has period 4π and $e^{-i\omega\frac{M}{2}}$ is also "periodic" with period 4π, but they line up in such a way that their product has period 2π.