EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 3
Department of Electrical Engineering and Computer Sciences 26 April 2007
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name Decimator FIRST Name Professor

Lab Time Anytime

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except three double-sided 8.5" $\times$ 11" sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 10.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the ten numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

Basic Formulas:

Discrete Fourier Series (DFS) Complex exponential Fourier series synthesis and analysis equations for a periodic discrete-time signal having period p:

$$x(n) = \sum_{k=\langle p \rangle} X_k \, e^{ik\omega_0 n} \qquad \longleftrightarrow \qquad X_k = \frac{1}{p} \sum_{n=\langle p \rangle} x(n) \, e^{-ik\omega_0 n} \ ,$$

where $\omega_0 = \dfrac{2\pi}{p}$ and $\langle p \rangle$ denotes a suitable discrete interval of length p (i.e., an interval containing p contiguous integers). For example, $\sum_{k=\langle p \rangle}$ may denote $\sum_{k=0}^{p-1}$ or $\sum_{k=1}^{p}$.

Continuous-Time Complex-Exponential Fourier Series Complex exponential Fourier series synthesis and analysis equations for a periodic continuous-time signal having period p:

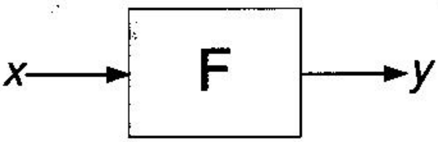
$$x(t) = \sum_{k=-\infty}^{\infty} X_k \, e^{ik\omega_0 t} \qquad \longleftrightarrow \qquad X_k = \frac{1}{p} \int_{\langle p \rangle} x(t) \, e^{-ik\omega_0 t} dt \ ,$$

where $\omega_0 = \dfrac{2\pi}{p}$ and $\langle p \rangle$ denotes a suitable continuous interval of length p. For example, $\int_{\langle p \rangle}$ can denote $\int_0^p$ or $\int_{-p/2}^{+p/2}$.

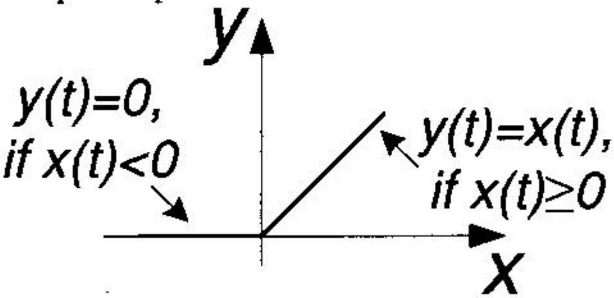
Continuous-Time Trigonometric Fourier Series Trigonometric Fourier series synthesis and analysis equations for a periodic continuous-time signal having period p (and fundamental frequency $\omega_0 = 2\pi/p$):

$$\begin{aligned}
x(t) &= A_0 + \sum_{k=1}^{+\infty} A_k \cos k\omega_0 t + \sum_{\ell=1}^{+\infty} B_\ell \sin \ell\omega_0 t \\
A_0 &= \frac{1}{p} \int_{\langle p \rangle} x(t) dt. \\
A_k &= \frac{2}{p} \int_{\langle p \rangle} x(t) \cos k\omega_0 t \, dt, \quad 1 \le k. \\
B_\ell &= \frac{2}{p} \int_{\langle p \rangle} x(t) \sin \ell\omega_0 t \, dt, \quad 1 \le \ell.
\end{aligned}$$

MT3.1 (35 Points) Consider a continuous-time system $\mathsf{F} : [\mathbb{R} \to \mathbb{R}] \to [\mathbb{R} \to \mathbb{R}]$ having input signal x and output signal y:



The system has the input-output characteristics shown below:

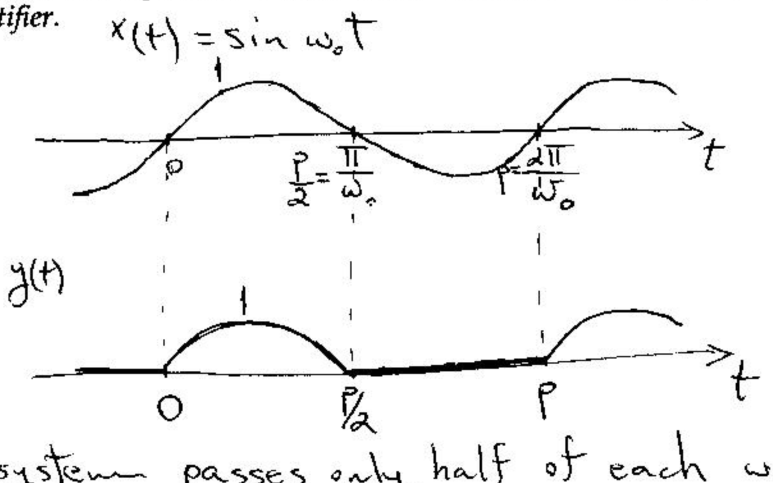


Suppose the input to the system is a periodic sinusoid having fundamental period p and characterized by

$$\forall t \in \mathbb{R}, \quad x(t) = \sin \omega_0 t,$$

where $\omega_0 = 2\pi/p$ is the fundamental frequency of the input sinusoid.

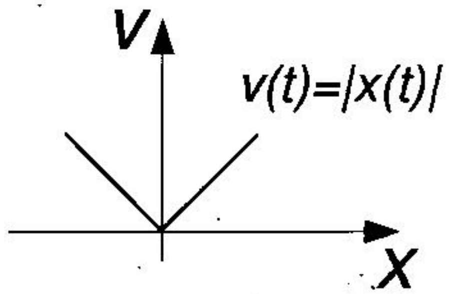
(a) Provide well-labeled plots of the input signal x *and* the output signal y over at least three periods. Explain why the system F is aptly called a *half-wave rectifier*.



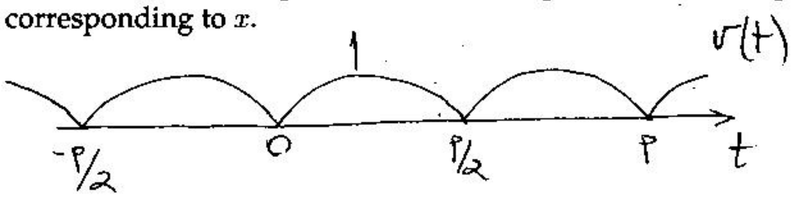
The system passes only half of each wave cycle and zeroes out the other half ⇒ It is called a half-wave rectifier.

(b) Suppose the input sinusoid x of Part (a) (i.e., $x(t) = \sin \omega_0 t$) is applied to another system G

which has the input-output characteristics shown below:



(i) Provide a well-labeled plot of at least three periods of the output signal v corresponding to x.



(ii) Assume the output signal y of system F (Part (a)) corresponding to the input sinusoid x has the following trigonometric Fourier series expansion:

$$y(t) = \frac{1}{\pi} + \frac{1}{2}\sin \omega_0 t - \frac{2}{\pi} \sum_{k=2,4,6,\ldots}^{+\infty} \frac{\cos k\omega_0 t}{k^2 - 1}.$$

Based on what you know about y, determine the trigonometric Fourier series expansion for the output signal v of system G. Hint: Notice that $v(t) = y(t) + y(t - p/2)$, and use this to advantage.

$$v(t) = y(t) + y(t - P/2)$$

$$y(t - P/2) = y\left(t - \frac{\pi}{\omega_0}\right) = \frac{1}{\pi} + \frac{1}{2}\sin\left[\omega_0\left(t - \frac{\pi}{\omega_0}\right)\right] - \frac{2}{\pi}\sum_{k=2,4,6,\ldots}^{+\infty} \frac{\cos k\omega_0\left(t - \frac{\pi}{\omega_0}\right)}{k^2-1}$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin(\omega_0 t - \pi) - \frac{2}{\pi}\sum_{k=2,4,6,\ldots} \frac{\cos(k\omega_0 t - k\pi)}{k^2-1}$$

Note:

$$\cos(k\omega_0 t - k\pi) = (-1)^k \cos k\omega_0 t = \cos k\omega_0 t \text{ if } k = \text{even}$$

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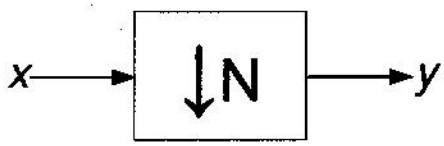
$$y(t - P/2) = \frac{1}{\pi} - \frac{1}{2}\sin \omega_0 t - \frac{2}{\pi}\sum_{k=2,4,6,\ldots}^{+\infty} \frac{\cos k\omega_0 t}{k^2-1}$$

$$\boxed{v(t) = y(t) + y(t - P/2) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=2,4,6,\ldots}^{+\infty} \frac{\cos k\omega_0 t}{k^2-1}}$$

Note: $v(t) = v(-t) \Rightarrow$ no sine terms in the expansion.

MT3.2 (35 Points) The figure below shows a discrete-time system that is variously called a *downsampler, sampling rate compressor*, or *decimator*.



The input and output signals are related as follows:

$$\forall n \in \mathbb{Z}, \quad y(n) = x(nN),$$

where $N \in \{2, 3, 4, \ldots\}$.

Suppose the input signal x is periodic and that its *fundamental* period is $p_x \in \mathbb{N}$.

(a) Show that the output signal y is periodic. What is the maximum value that the *fundamental* period p_y of the output signal y can have? Your answer should be in terms of p_x.

we want this

$$y(n+p_y) = x((n+p_y)N) = x(nN + p_y N) \stackrel{?}{=} x(nN) = y(n)$$

Must have $p_y N = k p_x \quad \exists k \in \mathbb{N} \implies p_y = \frac{k}{N} p_x$

The smallest k for which $\frac{k}{N}$ is an integer yields p_y.

The largest k can be is $k = N$, which yields $p_y = p_x$.

(b) Suppose p_x and N are coprime, i.e., their greatest common divisor $\gcd(p_x, N) = 1$; an example of a pair of coprime numbers is $(p_x = 3, N = 7)$, where $\gcd(3, 7) = 1$.

Determine the simplest possible expression for p_y. Your answer may be in terms of p_x, N, neither, or both, but it should *not* be particular to the numerical example $(3, 7)$ above. Do feel free, however, to let the particular example guide you to the general form of the answer.

k must be N

If p_x and N are coprime $\implies p_y = \frac{N}{N} p_x = p_x$

Examples: $p_x = 3, N = 7 \quad p_y = \frac{k}{7}(3) \implies k = 7$, so $p_y = 3 = p_x$

$p_x = 7, N = 3 \implies p_y = \frac{k}{3}(7) \implies k = 3$, so $p_y = 7 = p_x$

(c) Suppose N is divisible by p_x, i.e., there is a number $L \in \mathbb{N}$ such that $N = Lp_x$; an example is $(p_x = 2, N = 10)$. Determine p_y as specifically as possible. Your answer should *not* be particular to the numerical example $(2, 10)$. Do feel free, however, to let the example guide you to the general form of the answer.

$$N = L p_x \Longrightarrow p_y = \frac{k \cancel{p_x}}{L \cancel{p_x}} \Longrightarrow k = L, \text{ so } p_y = 1$$

(d) Suppose p_x is divisible by N, i.e., there is a number $M \in \mathbb{N}$ such that $p_x = MN$; an example is $(p_x = 8, N = 4)$. Determine p_y as specifically as possible. Your answer should *not* be particular to the numerical example $(8, 4)$. Do feel free, however, to let the example guide you to the general form of the answer.

$p_x = MN \Longrightarrow p_y = \frac{k}{\cancel{N}} M\cancel{N}$ already an integer $\Longrightarrow$

$k = 1 \Longrightarrow p_y = M$

(e) More generally, suppose p_x and N are *not* coprime, i.e., $\gcd(p_x, N) > 1$; an example of a pair of non-coprime numbers is $(p_x = 4, N = 6)$, where $\gcd(4, 6) = 2$.

Determine a simple expression for p_y. Your answer may be in terms of p_x, N, neither, or both, but *not* be particular to the numerical example $(4, 6)$. Do feel free, however, to let the particular example guide you to the general form of the answer.

Note: $\frac{p_x}{\gcd(p_x, N)} \in \mathbb{N}$. Now look at $p_y = \frac{k p_x}{N} = k \frac{\frac{p_x}{\gcd(p_x, N)}}{\frac{N}{\gcd(p_x, N)}} \Longrightarrow$

$\frac{N}{\gcd(p_x, N)} \in \mathbb{N}$ choose $k = \frac{N}{\gcd(p_x, N)} \Longrightarrow$

$$p_y = \frac{p_x}{\gcd(p_x, N)}$$

$p_x = ab$
$N = ac$
$\text{lcm}(p_x, N) = abc$

Alternatively, note $\text{lcm}(p_x, N)\gcd(p_x, N) = p_x N \Longrightarrow$

$$\frac{p_x N}{N \gcd(p_x, N)} = \frac{p_x}{\gcd(p_x, N)} = \frac{\text{lcm}(p_x, N)}{N} \Longrightarrow p_y = \frac{\text{lcm}(p_x, N)}{N}$$

MT3.3 (35 Points) The Hartley Series (HS) is an alternative orthogonal function expansion for periodic signals. If $x : \mathbb{R} \to \mathbb{R}$ is a real-valued periodic continuous-time signal having fundamental period p (and fundamental frequency $\omega_0 = 2\pi/p$), then its Hartley Series representation is given by:

$$x(t) = \sum_{-\infty}^{+\infty} C_k \operatorname{cas} k\omega_0 t,$$

where

$$\underbrace{\operatorname{cas} k\omega_0 t}_{\Gamma_k(t)} \triangleq \underbrace{\cos k\omega_0 t}_{\chi_k(t)} + \underbrace{\sin k\omega_0 t}_{\psi_k(t)}$$

is called the k^{th} *cosine and sine function*. The functions χ_k and ψ_k are the orthogonal functions in the trigonometric Fourier series expansion of x, and we know the following about them:

$$\begin{aligned}
\langle \chi_0, \chi_0 \rangle &= p \\
\langle \chi_k, \chi_k \rangle &= \langle \psi_k, \psi_k \rangle = \frac{p}{2} \text{ where } k \neq 0. \\
\langle \chi_k, \chi_m \rangle &= \langle \psi_k, \psi_m \rangle = 0 \text{ where } k \neq m. \\
\langle \chi_k, \psi_\ell \rangle &= 0 \text{ where } k, \ell \in \mathbb{Z}.
\end{aligned}$$

(a) Using the following definition for the inner product of two continuous-time signals of periodicity p,

$$\langle f, g \rangle = \int_{\langle p \rangle} f(t)\, g^*(t)\, dt,$$

(i) show that $\langle \Gamma_k, \Gamma_k \rangle = p, \forall k \in \mathbb{Z}$;

(ii) and establish that the cas functions are mutually orthogonal by showing that $\langle \Gamma_k, \Gamma_l \rangle = 0, \forall k, l \in \mathbb{Z}$, and $k \neq l$.

i) $\langle \Gamma_0, \Gamma_0 \rangle = \langle \chi_0, \chi_0 \rangle = p$

For $k \neq 0$, $\langle \Gamma_k, \Gamma_k \rangle = \langle \chi_k + \psi_k, \chi_k + \psi_k \rangle$

$= \langle \chi_k, \chi_k \rangle + \langle \chi_k, \psi_k \rangle + \langle \psi_k, \chi_k \rangle + \langle \psi_k, \psi_k \rangle$

$= \langle \chi_k, \chi_k \rangle + \langle \psi_k, \psi_k \rangle = \frac{p}{2} + \frac{p}{2} = p$

$\longrightarrow \langle \Gamma_k, \Gamma_k \rangle = p, \ \forall k \in \mathbb{Z}$

ii) $\langle \Gamma_k, \Gamma_l \rangle = \langle \chi_k + \psi_k, \chi_l + \psi_l \rangle$

$= \langle \chi_k, \chi_l \rangle + \langle \chi_k, \psi_l \rangle + \langle \psi_k, \chi_l \rangle + \langle \psi_k, \psi_l \rangle$

For $\forall k, l \in \mathbb{Z}$, $k \neq l$,

$\langle \Gamma_k, \Gamma_l \rangle = 0$

(b) Determine the analysis equation in which C_k is expressed in terms of the signal x and the k^{th} cas function Γ_k.

$$\langle x, \Gamma_k \rangle = \langle \sum_l C_l \Gamma_l , \Gamma_k \rangle$$

$$= \sum_l C_l \langle \Gamma_l , \Gamma_k \rangle$$

since $\langle \Gamma_l , \Gamma_k \rangle = 0$ for $k \neq l$,

$$\langle x, \Gamma_k \rangle = C_k \langle \Gamma_k , \Gamma_k \rangle$$

$$\rightarrow C_k = \frac{\langle x, \Gamma_k \rangle}{\langle \Gamma_k , \Gamma_k \rangle} = \frac{1}{p} \int_{\langle p \rangle} x(t) \operatorname{cas} k\omega_0 t \, dt$$

(c) If the complex-exponential Fourier series expansion of a *real-valued* continuous-time periodic signal x is given by

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k \, e^{ik\omega_0 kt},$$

express the HS coefficients C_k in terms of the real and imaginary parts of the coefficients X_k of the complex-exponential FS expansion of x.

From the HS analysis equation,

$$C_k = \frac{1}{p} \int_{\langle p \rangle} x(t) \operatorname{cas} k\omega_0 t \, dt = \frac{1}{p} \int_{\langle p \rangle} x(t) \cos k\omega_0 t \, dt + \frac{1}{p} \int_{\langle p \rangle} x(t) \sin k\omega_0 t \, dt$$

For $k = 0$,

$$C_0 = \frac{1}{p} \int_{\langle p \rangle} x(t) \, dt = A_0 = X_0$$

For $k \neq 0$,

we recognize that $A_k = \frac{2}{p} \int_{\langle p \rangle} x(t) \cos k\omega_0 t \, dt = 2 \operatorname{Re}\{X_k\}$

$$B_k = \frac{2}{p} \int_{\langle p \rangle} x(t) \sin k\omega_0 t \, dt = -2 \operatorname{Im}\{X_k\}$$

This results in the simplification

$$C_k = \frac{1}{2} A_k + \frac{1}{2} B_k = \operatorname{Re}\{X_k\} - \operatorname{Im}\{X_k\}$$

You may use this page for scratch work only.
Without exception, subject matter on this page will *not* be graded.

LAST Name Decimator FIRST Name Professor

Lab Time Any time

Problem	Points	Your Score
Name	10	10
1	35	35
3	35	35
3	35	35
Total	**115**	115