① $y[n] + \frac{1}{2} y[n-1] = x[n] - \frac{1}{2} x[n-1]$

a) $h[n] = ??$

$$h[n] + \frac{1}{2} h[n-1] = \delta[n] - \frac{1}{2} \delta[n-1] \quad (*)$$

$h[-1] = \underline{0}$; $\quad h[0] = \delta[0] - \frac{1}{2}\delta[-1] - \frac{1}{2} h[-1] = 1 - 0 - 0 = \underline{1}$

$h[1] = \delta[1] - \frac{1}{2}\delta[0] - \frac{1}{2} h[0] = 0 - \frac{1}{2} - \frac{1}{2} = \underline{-1}$

$h[2] = \delta[2] - \frac{1}{2}\delta[1] - \frac{1}{2} h[1] = 0 - 0 - (-\frac{1}{2}) = \underline{\frac{1}{2}}$

$h[3] = -\frac{1}{2} h[2] = \underline{-\frac{1}{4}}$ $\qquad h[4] = -\frac{1}{2} h[3] = \underline{1/8}$

$h[n] = 2(-\frac{1}{2})^n u[n] - \delta[n]$ fits pattern!

$\quad = \delta[n] - (-\frac{1}{2})^{n-1} u[n-1]$ etc, etc

b) $H(\omega) = ??$: $\quad x[n] = e^{i\omega n} \to y[n] = H(\omega) e^{i\omega n}$

$$H(\omega)e^{i\omega n} + \frac{1}{2} H(\omega) e^{i\omega n} e^{-i\omega} = e^{i\omega n} - \frac{1}{2} e^{i\omega n} e^{-i\omega}$$

$$H(\omega)\left[1 + \frac{1}{2} e^{-i\omega}\right] e^{i\omega n} = \left[1 - \frac{1}{2} e^{-i\omega}\right] e^{i\omega n}$$

$$H(\omega) = \frac{1 - \frac{1}{2} e^{-i\omega}}{1 + \frac{1}{2} e^{-i\omega}}$$

$$= \frac{2e^{i\omega} - 1}{2e^{i\omega} + 1}$$

① c) i) $x[n] = \left(\frac{1}{2}\right)^n u[n] \rightarrow X(\omega) = \frac{1}{1-\frac{1}{2}e^{-i\omega}}$

$$Y(\omega) = X(\omega)H(\omega) = \frac{1}{1-\frac{1}{2}e^{-i\omega}} \frac{1-\frac{1}{2}e^{-i\omega}}{1+\frac{1}{2}e^{-i\omega}}$$

$$= \frac{1}{1+\frac{1}{2}e^{-i\omega}} \rightarrow \boxed{y[n] = \left(-\frac{1}{2}\right)^n u[n]}$$

ii) $x[n] = \delta[n-2] + \frac{1}{2}\delta[n-3] \rightarrow y[n] = h[n-2] + \frac{1}{2}h[n-3]$

From (*) in (a), $h[n] + \frac{1}{2}h[n-2] = \delta[n] - \frac{1}{2}\delta[n-1]$

$$h[n-2] + \frac{1}{2}h[n-3] = \delta[n-2] - \frac{1}{2}\delta[n-3]$$

$$\boxed{y[n] = \delta[n-2] - \frac{1}{2}\delta[n-3]}$$

iii) $x[n] = \cos(\pi n) + i^n$

$$= e^{i\pi n} + e^{i\frac{\pi}{2}n}$$

$$y[n] = H(\pi)e^{i\pi n} + H\left(\frac{\pi}{2}\right)e^{i\frac{\pi}{2}n}$$

$$H(\pi) = \frac{2e^{-i\pi}-1}{2e^{-i\pi}+1} = \frac{-2-1}{-2+1} = 3$$

$$H\left(\frac{\pi}{2}\right) = \frac{2e^{i\pi/2}-1}{2e^{i\pi/2}+1} = \frac{-1+j2}{1+j2} = \frac{(-1+j2)(1-j2)}{1^2+2^2} = \frac{1}{5}(3+j4)$$

$$\boxed{y[n] = 3\cos(\pi n) + \frac{1}{5}(3+j4)\, i^n}$$

MT1.2 (10 Points) Consider a discrete-time signal $x: Z \to R$ that satisfies:

I. For every $n, l \in Z$, $x(n+4l) = x(n)$.

II. $\sum_{n=-1}^{2} x(n) = 2$.

III. $\sum_{n=-1}^{2} (-1)^n x(n) = 1$.

IV. $\sum_{n=-1}^{2} x(n)\cos(n\pi/2) = \sum_{n=-1}^{2} x(n)\sin(n\pi/2) = 0$.

(a) Determine the complex exponential Fourier series coefficients X_{-1}, X_0, X_1 and X_2 for this signal.

DTFS analysis eqn.: $X_k = \frac{1}{4}\sum_{n=-1}^{2} x(n) e^{-jk\omega_0 n}$, $\omega_0 = \frac{2\pi}{4}$

$$X_0 = \frac{1}{4}\sum_{n=-1}^{2} x(n)\, 1^n = \boxed{\frac{1}{2}}$$

$$X_2 = \frac{1}{4}\sum_{n=-1}^{2} x(n)(-1)^n = \boxed{\frac{1}{4}}$$

$$X_1 = \frac{1}{4}\sum_{n=-1}^{2} x(n)\, e^{-j\frac{\pi}{2}n} = \frac{1}{4}\sum_{n=-1}^{2} x(n)\left[\cos\frac{\pi}{2}n - j\sin\frac{\pi}{2}n\right] = \boxed{0}$$

Similarly, $\boxed{X_{-1} = 0}$

(b) Determine an expression for the signal x itself.

DTFS synthesis eqn: $x(n) = \sum_{k=-1}^{2} X_k e^{jk\omega_0 n}$

$$= \frac{1}{2}(1)^n + \frac{1}{4}(-1)^n$$

$$\boxed{x(n) = \frac{1}{2} + \frac{1}{4}(-1)^n}$$

MT1.3 (20 Points) Consider two discrete-time systems S_1 and S_2 that are described as follows. System S_1 has state s_1, inputs x_1, x_2 and output y and the following state-space model:

$$(S_1):\quad s_1(n+1) = As_1(n) + B_1x_1(n) + B_2x_2(n),\ y(n) = Cs_1(n),$$

and system S_2 has state s_2, input w and output p and the following state-space model:

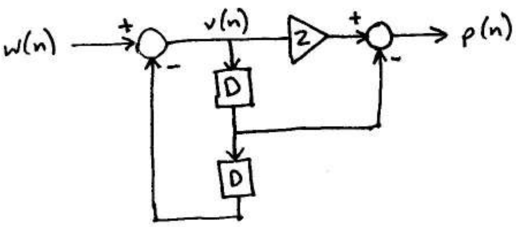
$$(S_2):\quad s_2(n+1) = Fs_2(n) + Gw(n),\ p(n) = Hs_2(n) + Jw(n),$$

In the models above, we do not specify the size of the matrices involved, but you can assume that they all have the correct dimensions (for example, G has row size given by the dimension of the state vector s_2).

(a) Suppose that the causal LTI system S_2 is characterized by the following linear, constant-coefficient difference equation (LCCDE):

$$p(n) + p(n-2) = 2w(n) - w(n-1).$$

Construct the state-space model for S_2 by finding the matrices F, G, H and J.



$v(n) = -v(n-2) + w(n)$

$p(n) = 2v(n) - v(n-1)$

$\quad = -v(n-1) - 2v(n-2) + 2w(n)$

Let $s_2(n) = \begin{bmatrix} v(n-1) \\ v(n-2) \end{bmatrix}$

$$\boxed{\begin{aligned} s_2(n+1) &= \begin{bmatrix} v(n) \\ v(n-1) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v(n-1) \\ v(n-2) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(n) \\ p(n) &= \begin{bmatrix} -1 & -2 \end{bmatrix}\begin{bmatrix} v(n-1) \\ v(n-2) \end{bmatrix} + 2w(n) \end{aligned}}$$

Alternative: $s_2(n) = \begin{bmatrix} p(n-1) \\ p(n-2) \\ w(n-1) \end{bmatrix}$

$s_2(n+1) = \begin{bmatrix} p(n) \\ p(n-1) \\ w(n) \end{bmatrix}$

$$s_2(n+1) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} p(n-1) \\ p(n-2) \\ w(n-1) \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} w(n)$$

$$p(n) = \begin{bmatrix} 0 & -1 & -1 \end{bmatrix} s_2(n) + 2w(n)$$

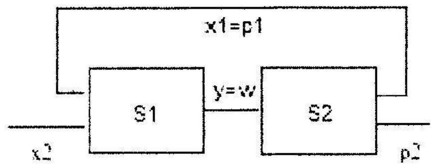


Figure 1: A connected system.

Now consider the block diagram shown in Figure 1, where $x_1 = p_1$ and $w = y$. Once again, system S_1 has state s_1, inputs x_1, x_2 and output y and the following state-space model:

$$(S_1):\quad s_1(n+1) = As_1(n) + B_1x_1(n) + B_2x_2(n),\ y(n) = Cs_1(n).$$

However, system S_2 now has state s_2, input w and **outputs p_1 and p_2** and the following state-space model:

$$(S_2):\quad s_2(n+1) = Fs_2(n) + Gw(n),\ p_1(n) = H_1s_2(n),\ p_2(n) = H_2s_2(n) + Jw(n),$$

For part (b), ignore the LCCDE representation and state-space model from part (a) completely – just leave your answer in terms of F, G, H_1, H_2 and J.

(b) Find a state-space model for this connected system, making sure to specify the state, inputs and outputs and all the system matrices. In other words, explicitly specify $\tilde{s}$, $\tilde{x}$ and $\tilde{y}$ as well as $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}$ satisfying

$$\tilde{s}(n+1) = \tilde{A}\tilde{s}(n) + \tilde{B}\tilde{x}(n),\ \tilde{y}(n) = \tilde{C}\tilde{s}(n) + \tilde{D}\tilde{x}(n).$$

$$\boxed{\tilde{s}(n) = \begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix} \qquad \tilde{x}(n) = x_2(n) \qquad \tilde{y}(n) = p_2(n)}$$

$x_1(n) = p_1(n) = H_1 s_2(n)$

$w(n) = y(n) = C s_1(n)$

$$s_1(n+1) = A s_1(n) + B_1 H_1 s_2(n) + B_2 x_2(n)$$

$$s_2(n+1) = F s_2(n) + G C s_1(n)$$

$$p_2(n) = H_2 s_2(n) + J C s_1(n)$$

$$\tilde{s}(n+1) = \begin{bmatrix} s_1(n+1) \\ s_2(n+1) \end{bmatrix} = \begin{bmatrix} A & B_1H_1 \\ GC & F \end{bmatrix} \begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix} + \begin{bmatrix} B_2 \\ 0 \end{bmatrix} x_2(n)$$

$$p_2(n) = \begin{bmatrix} JC & H_2 \end{bmatrix} \begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix} + 0\, x_2(n)$$

(4) a) $y[n] = \sum_{k=0}^{\infty} h(k)\, x(n-k)$, $x(k) = [x(0) \dots x(N)]$

Causality means $h(k) = 0 \;\; \forall\, k < 0$

$$y[n] = \sum_{k=0}^{\infty} h(k)\, x(n-k) = \sum_{k=0}^{\infty} x(k)\, h(n-k)$$

$$y[0] = x(0)h(0) + \cancelto{0}{x(1)h(-1)} + \cancelto{0}{x(2)h(-2)} + \cdots = x(0)h(0)$$

$$y[1] = x(0)h(1) + x(1)h(0) + \cancelto{0}{x(2)h(-1)} + \dots = x(1)h(0) + x(0)h(1)$$

$$y[2] = x(0)h(2) + x(1)h(1) + h(0)x(2)$$

$$\vdots$$

$$= \begin{bmatrix} h(2) & h(1) & h(0) & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} h(0) & 0 & 0 & 0 & \cdots & 0 \\ h(1) & h(0) & 0 & 0 & \cdots & 0 \\ h(2) & h(1) & h(0) & 0 & \cdots & 0 \\ \vdots & \vdots & & & \ddots & \\ h(N) & h(N-1) & h(N-2) & h(N-3) & \cdots & h(0) \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix}}_{X}$$

b) $x[k] = [x(-N), \dots, x(0)]$

$$y[0] = \sum_{k=0}^{\infty} h(k)\, x(0-k)$$
$$= h(0)x(0) + h(1)x(-1) + h(2)x(-2) + \cdots + h(N)x(-N)$$

$$y[1] = h(0)\cancelto{0}{x(1)} + h(1)x(0) + h(2)x(-1) + \dots + h(N+1)x(-N)$$

$$y[2] = h(2)x(0) + h(3)x(-1) + \dots + h(N+2)x(-N)$$

AND SO...

$$\underbrace{\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} h(N) & h(N-1) & \cdots & h(0) \\ h(N+1) & h(N) & \cdots & h(1) \\ h(N+2) & h(N+1) & \cdots & h(2) \\ \vdots & \vdots & & \vdots \\ h(2N) & h(2N-1) & \cdots & h(N) \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} x(-N) \\ x(-N+1) \\ \vdots \\ x(0) \end{bmatrix}}_{X}$$

MT1.5 (30 Points) Consider the equations

$$s_k(n+1) = a_k s_k(n) + b_k x(n),\ n \geq 0 .$$

This system is a simplistic model of the dynamics of prices of N financial assets, with $s_k(n)$ the rate of return of asset k at time n, and with x a signal that represents a common factor (say, the price of oil) affecting the entire market. In the equations above, $a_k \in R$ and $b_k \in R$ are given and are independent of time.

A given investment strategy is represented by a sequence w of vectors $w(n) \in R^N$, with $w_k(n)$ the relative amount invested in asset k at time n. We assume that $\forall n \geq 0$, $w_k(n) \geq 0$ for every $k = 1,\dots,N$ (we buy but do not sell), and that $w_1(n) + \cdots + w_N(n) = 1$ (so that $w_k(n)$ represents the proportion of wealth invested in asset k at time n). The quantity $y(n) := w(n)^T s(n)$ represents the rate of return of the portfolio at time n. In this problem, we focus on the resulting SISO system, which has x as input and y as output.

(a) Devise a state-space model for this SISO system, making sure to specify the state and all the system matrices.

$$s(n) = \begin{bmatrix} s_1(n) \\ s_2(n) \\ \vdots \\ s_N(n) \end{bmatrix} \qquad s(n+1) = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_N \end{bmatrix} s(n) + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} x(n)$$

$$y(n) = \begin{bmatrix} w_1(n) & w_2(n) & \cdots & w_N(n) \end{bmatrix} s(n) + 0\,x(n)$$

(b) Determine if the system has the following properties. For each property, prove that it holds (irrespective of the choice of the investment strategy w), or give a counterexample of a specific strategy that violates the property. You should assume that w is given a priori and not a function of the input.

(i) *Causal:* True. Assume input $x_1(n) = x_2(n)\ \forall n \leq n_0$. Given same initial condition, by recursion corresponding states $s_1(n) = s_2(n)\ \forall n \leq n_0$. $y_1(n) = w(n)^T s_1(n) = w(n)^T s_2(n) = y_2(n)\ \forall n \leq n_0$.

(ii) *Linear:* True. Let $\hat{x}(n) = a x_1(n) + b x_2(n)$. For initial condition zero, $\hat{s}(n+1) = A\hat{s}(n) + B\hat{x}(n)$ gives $\hat{s}(n) = a s_1(n) + b s_2(n)$ by recursion. $\hat{y}(n) = w(n)^T \hat{s}(n) = a y_1(n) + b y_2(n)\ \forall n \leq n_0$.

(iii) *Time-invariant:* False. Let $\hat{x}(n) = x(n-m)$. For zero initial condition, $\hat{s}(n+1) = A\hat{s}(n) + B\hat{x}(n)$ gives $\hat{s}(n) = s(n-m)$ by recursion. But $\hat{y}(n) = w(n)^T \hat{s}(n) = w(n)^T s(n-m) \neq w(n-m)^T s(n-m) = y(n-m)$.

From now on, we focus on a "buy-and-hold" strategy, where $w(n)$ is independent of n and is denoted simply by $w \in R^N$, with $w_k \geq 0$ for every $k = 1,\dots,N$ and $w_1 + \cdots + w_N = 1$.

(c) Determine the impulse response h of the system. Response when $x(n) = \delta(n)$

$n=0$: $s(1) = As(0) + B\delta(0) = As(0) + B$

$n \geq 1$: $s(n+1) = As(n) + B \cdot 0 = As(n)$

By recursion: $s(n) = A^n s(0) + A^{n-1} B$

$y(n) = Cs(n) + Dx(n)$

$= w^T s(n) + 0$

$$\boxed{y(n) = w^T \left[A^n s(0) + A^{n-1} B\right]} = \sum_{k=1}^{N} w_k \left[a_k^n s_k(0) + a_k^{n-1} b_k\right]$$

(d) Determine the frequency response H of the system.

DTFT: $e^{j\omega} S(e^{j\omega}) = AS(e^{j\omega}) + BX(e^{j\omega})$

$[Ie^{j\omega} - A] S(e^{j\omega}) = BX(e^{j\omega})$

$S(e^{j\omega}) = [Ie^{j\omega} - A]^{-1} BX(e^{j\omega})$

DTFT: $Y(e^{j\omega}) = CS(e^{j\omega}) + DX(e^{j\omega})$

$= C[Ie^{j\omega} - A]^{-1} BX(e^{j\omega})$

$$\boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = C[Ie^{j\omega} - A]^{-1} B} = \sum_{k=1}^{N} w_k \frac{1}{e^{j\omega} - a_k} b_k$$

(e) Determine the step response y_u of the system. $x(n) = u(n)$

$n=0$: $s_k(1) = a_k s_k(0) + b_k$

$n=1$: $s_k(2) = a_k s_k(1) + b_k$

$= a_k^2 s_k(0) + a_k b_k + b_k$

By recursion: $s_k(n) = a_k^n s_k(0) + b_k \sum_{p=0}^{n-1} a_k^p$

$= a_k^n s_k(0) + b_k \frac{1 - a_k^n}{1 - a_k}, \quad n \geq 0$

$$y(n) = w(n)^T s(n) = \boxed{\sum_{k=1}^{N} w_k \left[a_k^n s_k(0) + b_k \frac{1 - a_k^n}{1 - a_k}\right]}$$

(f) Select the strongest assertion, which is true, from the choices below. Explain your choice succinctly, but clearly and convincingly.

(i) The system is BIBO stable if $|a_k| < 1$ for every $k = 1,\dots,N$.

(ii) The system is BIBO stable if and only if $|a_k| < 1$ for every $k = 1,\dots,N$.

(iii) The system is BIBO stable if and only if for every $k = 1,\dots,N$, $|a_k| < 1$ or $b_k = 0$.

Since $y(n) = \sum_{k=1}^{N} w_k s_k(n)$, BIBO is equivalent to bounded input $\to$ bounded state.

if zero initial condition : (iii)
if not ZIC : (ii)

If $|a_k| < 1$, $\forall k = 1,\dots,N$, from (e):

$|s_k(n)| \le |a_k|^n |s(0)| + |b_k| \dfrac{1 + |a_k|^n}{1 - |a_k|}$

$|a_k|^n \le 1$, $\forall n \Rightarrow |s_k(n)| \le |s(0)| + 2\dfrac{|b_k|}{1 - |a_k|}$, which is bounded

or $b_k = 0 \Rightarrow s_k(n) = 0$ if $s(0) = 0$

Bounded input, bounded state $\to |a_k| < 1$, $\forall k = 1,\dots,N$ by contrapositive:

Let $|a_\ell| \ge 1$ for some $\ell \in \{1,\dots,N\}$

$s_\ell(n) = a_k^n s(0) + b_k \dfrac{1 - a_k^n}{1 - a_k}$ is unbounded unless $s(0) = 0$ or $b_k = 0$

Now assume that $|b_k| = 1$, $|a_k| < 1$ for every $k = 1,\dots,N$, and we define $\bar{a} = \max_{1 \le k \le N} |a_k|$.

(g) Show that if $\bar{a} \le \frac{2}{3}$, then for every sequence x such that $x(n) \in [-0.1, 0.1]$ for every n, we have $y(n) \in [-0.3, 0.3]$ for every n.

Assume $|s_k(0)| \le 0.3$ (For instance, zero initial conditions)

$$
\begin{aligned}
|s_k(n+1)| &\le |a_k||s_k(n)| + |b_k||x(n)| \\
&\le \bar{a}|s_k(n)| + 0.1 \\
&\le \tfrac{2}{3}|s_k(n)| + 0.1
\end{aligned}
$$

if $|s_k(n)| \le 0.3$:

$|s_k(n+1)| \le \frac{2}{3} 0.3 + 0.1 = 0.3$

By recursion, $|s_k(n)| \le 0.3$ $\forall k, n$

$$
\begin{aligned}
|y(n)| &= \sum_{k=1}^{N} |w_k||s_k(n)| \\
&\le 0.3 \sum_{k=1}^{N} |w_k| = 0.3 \quad \forall n
\end{aligned}
$$